An Efficient TFETI Based Solver for Elasto-Plastic Problems of Mechanics

Martin CERMAK[1], *Tomas KOZUBEK*[1]

[1]Department of Applied Mathematics, Faculty of Electrical Engineering and Computer Science, VSB–Technical University of Ostrava, 17. listopadu 15, 708 33 Ostrava Poruba, Czech Republic

martin.cermak@vsb.cz, tomas.kozubek@vsb.cz

Abstract. *This paper illustrates how to implement effective solvers for elasto-plastic problems. We consider the time step problems formulated by nonlinear variational equations in terms of displacements. To treat nonlinearity and nonsmoothness we use semismooth Newton method. In each Newton iteration we have to solve linear system of algebraic equations and for its numerical solution we use TFETI algorithm. In our benchmark we demonstrate our approach on von Mises plasticity with isotropic hardening and use return mapping concept.*

Keywords

Domain decomposition, elasto-plasticity, TFETI.

1. Introduction

The paper is organized as follows. We briefly review the TFETI methodology that transforms the large primal problem of elastostatics in terms of displacements into the smaller and better conditioned dual one in terms of the Lagrange multipliers (pressures) whose conditioning is further improved by using the projectors defined by the natural coarse grid. Further we briefly review the elasto-plasticity methodology for von Mises plasticity with isotropic hardening. We illustrate the efficiency of our algorithm on the solution of 3D elasto-plastic model benchmark and give encouraging results of numerical experiments.

2. Problem of Elastostatics

Let us consider an isotropic elastic body represented in a reference configuration by a domain Ω in $\mathbb{R}^d, d = 2,3$, with the sufficiently smooth boundary Γ as in Fig. 1. Suppose that Γ consists of two disjoint parts Γ_U and Γ_F, $\Gamma = \overline{\Gamma}_U \cup \overline{\Gamma}_F$, and that the displacements $\mathbf{U}: \Gamma_U \to \mathbb{R}^d$ and forces $\mathbf{F}: \Gamma_F \to \mathbb{R}^d$ are given. The mechanical properties of Ω are defined by the Young modulus E, the Poisson ratio ν, and the density ρ.

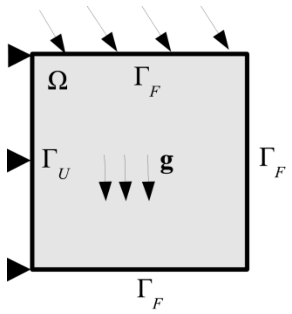

Fig. 1: Model problem.

Let $\mathbf{C}: \Omega \to \mathbb{R}^{2\times2\times2\times2}_{sym}$, $\langle \mathbf{C}\tau, \sigma \rangle = \langle \tau, \mathbf{C}\sigma \rangle \, \forall \tau, \sigma \in \mathbb{R}^{2\times2}_{sym}$, $c_{ijkl} = c_{ijlk} = c_{klij}$, where $c_{ijkl}: \Omega \to \mathbb{R}$ and $\mathbf{g}: \Omega \to \mathbb{R}^d$ denote the component of the elasticity tensor $\mathbf{C}$ and a vector of body forces, respectively. For any sufficiently smooth displacement $\mathbf{u}: \overline{\Omega} \to \mathbb{R}^d$, the total potential energy is defined by

$$J(\mathbf{u}) = \frac{1}{2} a(\mathbf{u},\mathbf{u}) - \int_\Omega \mathbf{g}^\top \mathbf{u} \, d\Omega - \int_{\Gamma_F} \mathbf{F}^\top \mathbf{u} \, d\Gamma, \qquad (1)$$

where

$$\mathrm{a}(\mathbf{u},\mathbf{v}) = \int_\Omega \mathrm{c}_{\mathrm{ijkl}} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{kl}(\mathbf{v}) \, d\Omega, \qquad (2)$$

and

$$\varepsilon_{\mathrm{kl}}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right).$$

We suppose that the elasticity tensor satisfies natural physical restrictions so that

$$a(\mathbf{u},\mathbf{v}) = a(\mathbf{v},\mathbf{u}) \text{ and } a(\mathbf{u},\mathbf{u}) \ge 0. \qquad (3)$$

Now let us introduce the Sobolev space

$V = H^1(\Omega)^d$ and let

$$K = \{\mathbf{v} \in V : \mathbf{v} = \mathbf{U} \text{ on } \Gamma_U\}.$$

be its non-empty, if there exists a function $\mathbf{u}_0 \in \left[H^1(\Omega)\right]^d$ such that $\mathbf{u}_0\big|_{\Gamma_U} = \mathbf{U}$, closed, and convex subset. The displacement $\mathbf{u} \in K$ of body in equilibrium satisfies

$$J(\mathbf{u}) \le J(\mathbf{v}) \text{ for any } \mathbf{v} \in K. \tag{4}$$

Conditions that guarantee existence and uniqueness may be expressed in terms of coercivity of J. More general boundary conditions, such as prescribed normal displacements and periodicity, may be considered without any conceptual difficulties.

3. TFETI Domain Decomposition

To apply the TFETI domain decomposition, we tear the body from the part of the boundary with the Dirichlet boundary condition, decompose the body into subdomains, assign each subdomain a unique number, and introduce new “gluing” conditions on the artificial intersubdomain boundaries and on the boundaries with imposed Dirichlet data.

More specifically, the original body Ω is decomposed into a system of s homogeneous isotropic elastic bodies, each of which occupies, in a reference configuration, a subdomain Ω^p in $\mathbb{R}^d, d = 2, 3$, $p = 1, \ldots, s$. After decomposition each boundary Γ^p of Ω^p consists of three disjoint parts Γ_U^p, Γ_F^p, and Γ_G^p, $\Gamma^p = \overline{\Gamma}_U^p \cup \overline{\Gamma}_F^p \cup \overline{\Gamma}_G^p$, with the corresponding displacements $\mathbf{U}^p$ and forces $\mathbf{F}^p$ inherited from the originally imposed boundary conditions on Γ. For the artificial intersubdomain boundaries, we use the following notation: Γ_G^{pq} denotes the part of Γ^p that is glued to Ω^q and Γ_G^p denotes the part of Γ^p that is glued to the other subdomains. Obviously $\Gamma_G^{pq} = \Gamma_G^{qp}$. An auxiliary decomposition of the problem with renumbered subdomains and artificial intersubdomain boundaries is in Fig. 2.

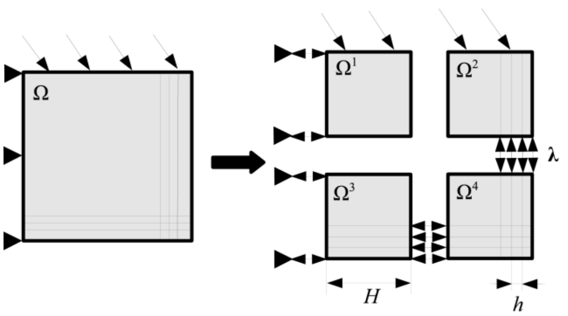

Fig. 2: TFETI domain decomposition with subdomain renumbering and traces of discretization.

The gluing conditions require continuity of the displacements and of their normal derivatives across the intersubdomain boundaries. The mechanical properties of Ω^p are defined by the Young modulus E^p, the Poisson ratio ν^p, and the density ρ^p.

Let $\mathbf{C}^p : \Omega^p \to \mathbb{R}_{sym}^{2\times2\times2\times2}$, $\langle \mathbf{C}^p \tau, \sigma \rangle = \langle \tau, \mathbf{C}^p \sigma \rangle \forall \tau, \sigma \in \mathbb{R}_{sym}^{2\times2}$, $c_{ijkl}^p = c_{ijlk}^p = c_{klij}^p$, where $c_{ijkl}^p : \Omega \to \mathbb{R}$ and $\mathbf{g}^p$ denote again the entries of the elasticity tensor and a vector of body forces, respectively. For any sufficiently smooth displacement $\mathbf{u} : \overline{\Omega}^1 \times \ldots \times \overline{\Omega}^s \to \mathbb{R}^d$, the total potential energy is defined by

$$J(\mathbf{u}) = \sum_{p=1}^{s} \left\{ \frac{1}{2} a^p(\mathbf{u}^p, \mathbf{u}^p) - \int_{\Omega^p} (\mathbf{g}^p)^\top \mathbf{u}^p d\Omega \right.$$

$$\left. - \int_{\Gamma_F^p} (\mathbf{F}^p)^\top \mathbf{u}^p d\Gamma \right\}, \tag{5}$$

where

$$a^p\left(\mathbf{u}^p, \mathbf{v}^p\right) = \int_{\Omega} c_{ijkl}^p \varepsilon_{ij}\left(\mathbf{u}^p\right) \varepsilon_{kl}^p\left(\mathbf{v}^p\right) d\Omega, \tag{6}$$

and

$$\varepsilon_{kl}^p\left(\mathbf{u}^p\right) = \frac{1}{2}\left(\frac{\partial u_k^p}{\partial x_l^p} + \frac{\partial u_l^p}{\partial x_k^p}\right).$$

Let us introduce the product Sobolev space

$$\mathbf{V} = H^1(\Omega^1)^d \times \ldots \times H^1(\Omega^s)^d, \tag{7}$$

and let

$$\kappa = \left\{ \mathbf{v} = \left(\mathbf{v}^1, \ldots, \mathbf{v}^s\right) \in \mathbf{V} : \mathbf{v}^p - \mathbf{U}^p \text{on } \Gamma_U^p, v^p = v^q \text{on } \Gamma_G^{pq} \right\}$$

be its non-empty, closed, and convex subset. The displacement $\mathbf{u} \in \kappa$ of the system of subdomains in equilibrium satisfies

$$J\left(\mathbf{u}\right) \le J\left(\mathbf{v}\right) \text{ for any } \mathbf{v} \in \kappa. \tag{8}$$

The finite element discretization of $\overline{\Omega} = \overline{\Omega}^1 \cup \ldots \cup \overline{\Omega}^s$ with a suitable numbering of nodes results in the quadratic programming (QP) problem

$$\min_{\mathbf{u}} \frac{1}{2} \mathbf{u}^\top \mathbf{K} \mathbf{u} - \mathbf{f}^\top \mathbf{u} \quad \text{subject to } \mathbf{B}\mathbf{u} = \mathbf{c}, \tag{9}$$

where $\mathbf{K} = \mathrm{diag}(\mathbf{K}_1, \ldots, \mathbf{K}_s)$ denotes a symmetric positive semidefinite block-diagonal stiffness matrix of order n, $\mathbf{B}$ denotes an $m \times n$ full rank constraint matrix, $\mathbf{f} \in \mathbb{R}^n$ is a load vector, and $\mathbf{c} \in \mathbb{R}^m$ is a constraint vector.

The algorithm for solving the minimization problem (9) can found in [4].

The diagonal blocks $\mathbf{K}_p$ that correspond to the subdomains Ω^p are positive semidefinite sparse matrices with known kernels, the rigid body modes. This is a great advantage because all blocks can be effectively regularized and then decomposed using any standard sparse Cholesky type factorization method for nonsingular matrices [2], [4].

The matrix $\mathbf{B}$ with the rows $\mathbf{b}_i$ and the vector $\mathbf{c}$ with the entries c_i enforce the prescribed displacements on the part of the boundary with imposed Dirichlet condition and the continuity of the displacements across the auxiliary interfaces.

A parallel and numerically scalable algorithm for the numerical solution of (9) is introduced in [4] with scalability demonstrated up to 315 millions of unknowns and 4800 cores.

4. Elasto-Plasticity

Elasto-plastic problems are the so-called quasi-static problems where the history of loading is taken into account. We consider the von Mises elasto-plasticity with the strain isotropic hardening and incremental finite element method with the return mapping concept [1].

The elasto-plastic deformation of a body Ω after loading is described by the Cauchy stress tensor σ, the small strain tensor ε, the displacement $\mathbf{u}$, and the nonnegative hardening parameter κ. Symmetric tensor is represented by the vector and its deviatoric part is denoted by the symbol *dev*.

Let us denote the space of continuous and piecewise linear functions constructed over a regular triangulation of $\bar{\Omega}$ with the discretization norm h by $V_h \subset V$, where $V = \left\{ \mathbf{v} \in H^1(\Omega)^d : \ \mathbf{v} = 0 \text{ on } \Gamma_U \right\}$. Let

$$0 = t_0 < t_1 < \ldots t_k < \ldots < t_N = t^*, \tag{10}$$

be a partition of the time interval $[0, t^*]$. Then the solution algorithm after time and space discretizations has the form:

Algorithm 1:

1. Initial step: $\mathbf{u}_h^0 = 0,\ \sigma_h^0 = 0,\ \kappa_h^0 = 0$.
2. **for** $k = 0, \ldots, N-1$ **do** (load step).
3. From previous step we know: $u_h^k,\ \sigma_h^k,\ \kappa_h^k$ and compute $\Delta \mathbf{u}_h,\ \Delta\sigma_h,\ \Delta\kappa_h$

$$\Delta\varepsilon_h = \varepsilon\left(\Delta\mathbf{u}_h\right), \quad \Delta\mathbf{u}_h \in V_h, \tag{11}$$

$$\Delta\sigma_h = T_\sigma\left(\sigma_h^k, \kappa_h^k, \Delta\varepsilon_h\right), \tag{12}$$

$$\Delta\kappa_h = T_\kappa\left(\sigma_h^k, \kappa_h^k, \Delta\varepsilon_h\right). \tag{13}$$

4. Solution $\Delta\sigma_h\left(\sigma_h^k, \kappa_h^k, \varepsilon\left(\Delta\mathbf{u}_h\right)\right)$ is substituted into equation of equilibrium:

$$\begin{aligned}&\int_\Omega \left\langle \Delta\sigma\left(\sigma_h^k, \kappa_h^k, \varepsilon\left(\Delta\mathbf{u}_h\right)\right), \varepsilon\left(\mathbf{v}_h\right)\right\rangle dx = \\ &\left\langle \Delta\mathbf{f}_h^k, \mathbf{v}_h\right\rangle, \forall \mathbf{v}_h \in V_h\end{aligned}. \tag{14}$$

This leads to a nonlinear system of equations with unknown $\Delta\mathbf{u}_h$ which is solved using the Newton method. The linearized problem arising in each Newton step is solved by the TFETI algorithmic scheme [4], [8]. It is possible because the stiffness matrix of linearized system is symmetric and positive semidefinite.

5. Then we compute values for the next step:

$$\mathbf{u}_h^{k+1} = \mathbf{u}_h^k + \Delta\mathbf{u}_h, \sigma_h^{k+1} = \sigma_h^k + \Delta\sigma_h, \kappa_h^{k+1} = \kappa_h^k + \Delta\kappa_h.$$

6. **enddo**.

For return mapping concept we define operators T_σ^{RM} and T_κ^{RM}. Their form are $T_\sigma^{RM} = T_\sigma = \Delta\sigma_h$ and $T_\kappa^{RM} = T_\kappa = \Delta\kappa_h$ Now we can go from tensor notation $\sigma_h, \varepsilon_h, \kappa_h$ to the algebraic notation $\boldsymbol{\sigma}_h, \boldsymbol{\varepsilon}_h, \boldsymbol{\kappa}_h$ for stress, strain and hardening variables. Above we consider the following notation. Let $\mathbf{C}$ denote the Hook's matrix, $\mathbf{E}$ represent linear operator *dev*, μ, λ be the Lame coefficients, $\vartriangle\mathbf{f}_h^k$ be the increment of the right hand side, and $\boldsymbol{\sigma}_h^t = \boldsymbol{\sigma}_h^k + \mathbf{C}\vartriangle\boldsymbol{\varepsilon}_h$. Lets

$$\vartriangle\boldsymbol{\sigma}_h = \begin{cases} \mathbf{C}\vartriangle\boldsymbol{\varepsilon}_h & \text{if } \ P(\boldsymbol{\sigma}_h^t, \boldsymbol{\kappa}_h^k) \le 0, \\ \mathbf{C}\vartriangle\boldsymbol{\varepsilon}_h - \gamma_R \hat{\mathbf{n}} & \text{if } \ P(\boldsymbol{\sigma}_h^t, \boldsymbol{\kappa}_h^k) > 0, \end{cases} \tag{15}$$

$$\vartriangle\boldsymbol{\kappa}_h = \begin{cases} 0 & \text{if } \ P(\boldsymbol{\sigma}_h^t, \boldsymbol{\kappa}_h^k) \le 0, \\ \gamma z = \gamma_R \| \ \mathbf{Cp} \|^{-1} z & \text{if } \ P(\boldsymbol{\sigma}_h^t, \boldsymbol{\kappa}_h^k) > 0, \end{cases} \tag{16}$$

where

$$\gamma_R = \frac{3\mu}{3\mu + H_m}\sqrt{\frac{2}{3}}P(\boldsymbol{\sigma}_h^t, \boldsymbol{\kappa}_h^k), \tag{17}$$

$$\hat{\mathbf{n}} = \frac{dev(\boldsymbol{\sigma}_h^t)}{\| \ dev(\boldsymbol{\sigma}_h^t)\|}, \quad \| \ \mathbf{Cp}\| = 2\mu\sqrt{\frac{3}{2}}, \quad z = 1, \tag{18}$$

and plasticity function

$$P(\boldsymbol{\sigma}_h^t, \boldsymbol{\kappa}_h^k) = \sqrt{\frac{3}{2}}\| \ dev(\boldsymbol{\sigma}_h^t)\| \ -(Y + H_m\boldsymbol{\kappa}_h^k), \tag{19}$$

where $Y, H_m > 0$. The function $\gamma_R\hat{\mathbf{n}}$ is semismooth and potential, that's why we can solve this linearized system by algorithm T-FETI. We want to achieve quadratic convergence, and therefore we compute the derivative of. T_σ^{RM}. The form of T_σ^{RM} is

$$(T_\sigma^{RM})'(\vartriangle\boldsymbol{\varepsilon}) = \mathbf{C} - 2\mu\frac{3\mu}{3\mu + H_m}\left[\mathbf{E} + \sqrt{\frac{2}{3}}\frac{Y_0 + H_m\boldsymbol{\kappa}_h^k}{\| \ dev(\boldsymbol{\sigma}_h^k + \mathbf{C}\vartriangle\boldsymbol{\varepsilon})\|}\right.$$

$$\left(\frac{dev(\boldsymbol{\sigma}_h^k + \mathbf{C}\triangle\boldsymbol{\varepsilon})(dev(\boldsymbol{\sigma}_h^k + \mathbf{C}\triangle\boldsymbol{\varepsilon}))^T}{\| \; dev(\boldsymbol{\sigma}_h^k + \mathbf{C}\triangle\boldsymbol{\varepsilon})\|^2} - \mathbf{E}\right)\Bigg]. \quad (20)$$

If we represent a function $\mathbf{v}_h \in V_h$ by the vector $\mathbf{v} \in \mathbb{R}^n$ and omit index k then (14) can be rewritten as the system of nonlinear equations

$$F(\triangle\mathbf{u}) = \triangle\mathbf{f}\ , \quad (21)$$

where

$$\langle F(\mathbf{v}), \mathbf{w}\rangle = \int_\Omega \langle T_\sigma^{RM}(\varepsilon(\mathbf{v}_h)), \varepsilon(\mathbf{w}_h)\rangle dx, \quad \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

$$\langle \triangle\mathbf{f}, \mathbf{w}\rangle = \triangle\mathbf{f}_h(v_h), \quad \forall \mathbf{w} \in \mathbb{R}^n\ . \quad (22)$$

Similarly we build the tangent stiffness matrix

$$\langle \mathbf{Kv}, \mathbf{w}\rangle = \int_\Omega \Big\langle \left(T_\sigma^{RM}\right)'\left(\Delta\varepsilon\left(u_{k-1}\right)\varepsilon\left(v\right), \varepsilon\left(w\right)\right)\Big\rangle dx, \quad \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

where $\mathbf{u}_{k-1}$ is displacement from previous Newton step.

5. Numerical Experiments

Described algorithms were implemented in MatSol library [5] developed in Matlab environment and tested on the solution of 3D problems.

Let us consider a 3D plate with a hole in the center (due to symmetry only a quarter of the whole structure is used). The geometry of the body with traces of decomposition and discretization is depicted in Fig. 3. In Fig. 4 we see a zoom of Fig. 3 near the hole. Symmetry conditions are prescribed on the left and lower sides of Ω. The surface load $g(t) = 450\sin(2\pi t)$ [MPa], $t \in [0,1/4]$ [sec], is applied to the upper side of Ω. The elasto-plastic material parameters are E = 206900 [MPa], $\nu = 0.29$, $Y = 450[\text{MPa}]$, $H_m = 100[\text{MPa}]$ and the time interval $[0,1/4]$ [sec] is divided into 50 steps. We consider a mesh with 4450 nodes and 19008 tetrahedrons. A similar numerical example was also investigated in [3].

In the n-th Newton iteration we compute an approximation $\triangle\mathbf{u}^n$ by solving the constrained linear problem of the form

$$\min_{\mathbf{B}\triangle\mathbf{u}^n = \mathbf{o}} \frac{1}{2}\left(\triangle\mathbf{u}^n\right)^\top \mathbf{K}^n \triangle\mathbf{u}^n - \left(\triangle\mathbf{u}^n\right)^\top \triangle\mathbf{f}^n ,$$

using the scalable TFETI algorithmic scheme proposed in [4]. We stop the Newton method in every time step if $\| \triangle\mathbf{u}^{n+1} - \triangle\mathbf{u}^n\| \;/ \left(\| \triangle\mathbf{u}^{n+1}\| + \| \triangle\mathbf{u}^n\|\right)$ is less than 10^{-9}.

Notice that the maximum number of the Newton iterations is small for all time steps, therefore the method is suitable for the problem. In the following figures, we depict plastic and elastic elements and von Mises stress in the xy plane with the z coordinate 0 [mm]. In Fig. 5, 6, 7 and 8, we can see which elements are plastic (gray color) and which are elastic (white color) in chosen time steps. Particularly, in time steps 1-12 we observe only elastic behavior, and in time steps 13-50 plastic behavior of some elements. The maximum value of hardening at each time step is depicted in Fig. 9. The von Mises stress distribution on deformed mesh is showed in Fig. 10.

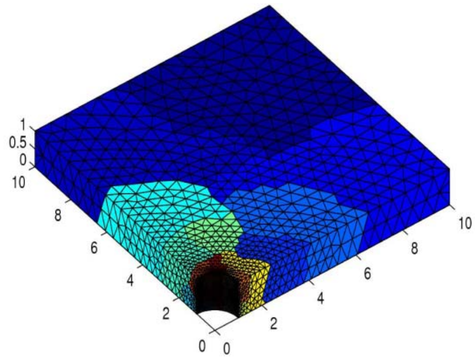

Fig. 3: Geometry in [mm] with traces of decomposition and discretization.

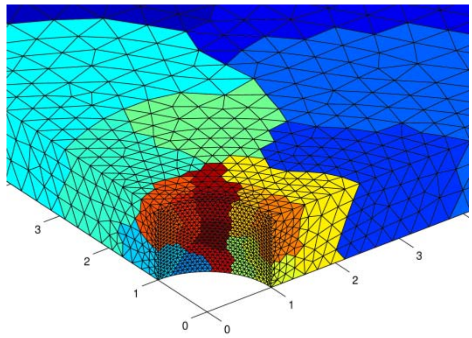

Fig. 4: Zoom of Fig. 3 near the hole.

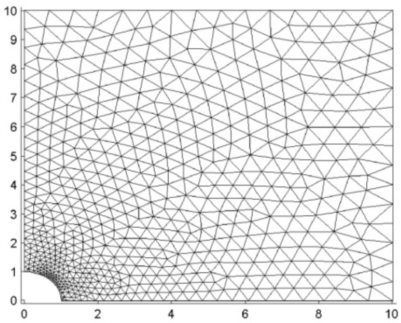

Fig. 5: Plastic and elastic elements after 1 time step.

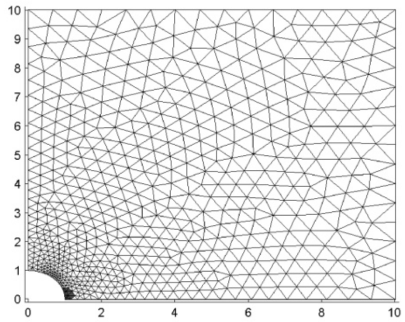

Fig. 6: Plastic and elastic elements after 20 time steps.

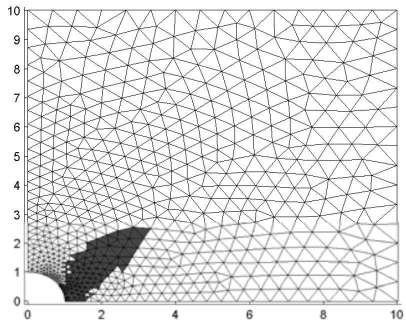

Fig. 7: Plastic and elastic elements after 35 time steps.

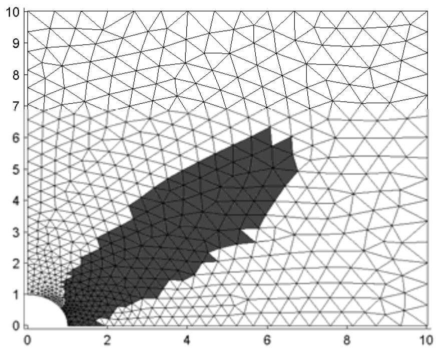

Fig. 8: Plastic and elastic elements after 50 time steps.

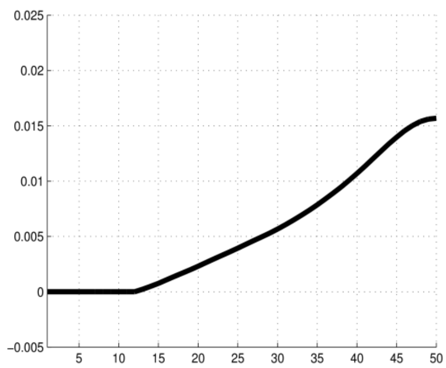

Fig. 9: Maximum values of hardening in time iterations.

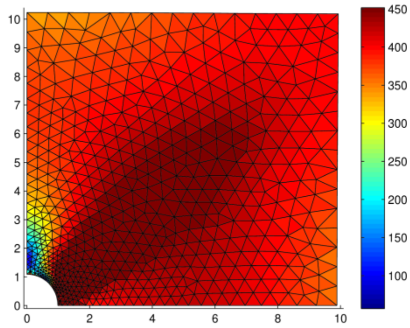

Fig. 10: Von Mises stress distribution on the deformed mesh (scaled 10x).

6. Conclusion and Goals

We have presented an efficient algorithm for the numerical solution of elasto-plastic problems. These problems lead to the quasi-static problems, where each nonlinear and nonsmooth time step problem is solved by the semismooth Newton method. In each Newton iteration we have to solve an auxiliary (possibly of large size) linear system of algebraic equations. We proposed a new approach how to solve such system efficiently using in a sense optimal algorithm based on our Total-FETI variant of FETI domain decomposition method. We illustrated the efficiency of our algorithm on the solution of 3D elasto-plastic model benchmark and gave results of numerical experiments. The results indicate that the algorithm may be efficient.

Nowadays we adapt this approach to the solution of contact problems.

Acknowledgements

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About Authors

Martin CERMAK was born in 1983 in Hranice, graduated from the Faculty of Electrical Engineering and Computer Science of the VSB-Technical University of Ostrava in 2008 in the field of Interface between COMSOL and OOSol for Solving Contact Problems. Currently he is a Ph.D. student at the Department of Applied Mathematics of the VSB-Technical University of Ostrava, a research assistant (Ph.D.) at the Centre of Excellence IT4innovations. His current research interests concern mainly the TFETI Domain decomposition and elasto-plastic problems for equality and inequality.

Tomas KOZUBEK was born in 1975 in Karvina, graduated from the Faculty of Electrical Engineering and Computer Science of the VSB-Technical University of Ostrava in 1998 in the field of Fictitious Domain Methods for the Numerical Solution of PDEs. He completed his Ph.D. studies in the field of Shape Optimization in 2002 and habilitated in the field of Numerical Solution of Partial Differential Equations using Wavelets and Fictitious Domain Approach in 2007. Currently he is an associate professor at the Department of Applied Mathematics of the VSB-Technical University of Ostrava, a research programme manager at the Centre of Excellence IT4innovations, and a main coordinator of the SPOMECH project focused on reliable solution of nonlinear problems in mechanics.