HAMILTON´S PRINCIPLE AND ELECTRIC CIRCUITS THEORY

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Summary In the theory of electrical or electromechanical circuits different methods are known for construction of mathematical model. In this paper another, alternative method is introduced that is based on Hamilton variational principle that is generally valid in physics.

1. EXAMPLE AS AN INTRODUCTION

The analysis of a capacity circuit according Fig. 1 is done in a non-traditional way based on the minimum energy principle. According to this principle the voltage in the branches of the circuit is distributed so that the energy of electric field is minimal.

In the circuit theory different topological qualities of the circuit are examined (see e.g. [6]). One of these qualities expresses *that introducing into the graph of circuit one of its trees and knowing the voltage on the passive branches of this tree, we can identify the voltage on the remaining branches of the circuit.* This knowledge is now applied on the circuit in Fig.1. Let us take a tree (v_1, v_2, v_6); in his passive branches there are voltages U_1, U_2 . With help of these voltages, applying the voltage Kirchhoff`s Law, we can easily define the remaining voltage U_3, U_4, U_5.

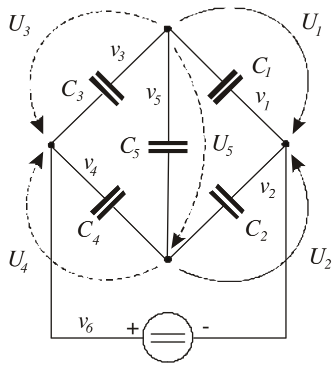

Fig. 1. To the example

The energy of the electric field of the whole circuit (i.e. the sum of energies of its capacitors) can be expressed using branch voltages U_1 and U_2:

$$W_e = U_1^2 + \frac{5}{2}U_2^2 + \frac{1}{2}(U_1 - 100)^2 + \frac{1}{2}(U_2 - 100)^2 + \frac{3}{2}(U_1 - U_2)^2 \tag{1}$$

For the numerical values of parameters of branch elements: C_1 = 2 μF, C_2 = 5 μF, C_3 = 1 μF, C_4 = 4 μF, C_5 = 3 μF, U_0 = 100V the expression for energy is

$$W_e = U_1^2 + \frac{5}{2}U_2^2 + \frac{1}{2}(U_1 - 100)^2 + \frac{1}{2}(U_2 - 100)^2 + \frac{3}{2}(U_1 - U_2)^2$$

Now the principle of minimum energy is used. The dependence $W_e = f(U_1, U_2)$ is expressed in Fig. 2. This function gets to its minimum at the point with the coordinates U_1 = 38,095 V and U_2 = 42,857 V. This branch voltages, together with remaining voltages $U_3 = U_1 - 100 = -61{,}095$ V, $U_4 = U_2 - 100 = -57{,}143$ V and $U_5 = U_1 - U_2 = -4{,}762$ V are the solution of the circuit.

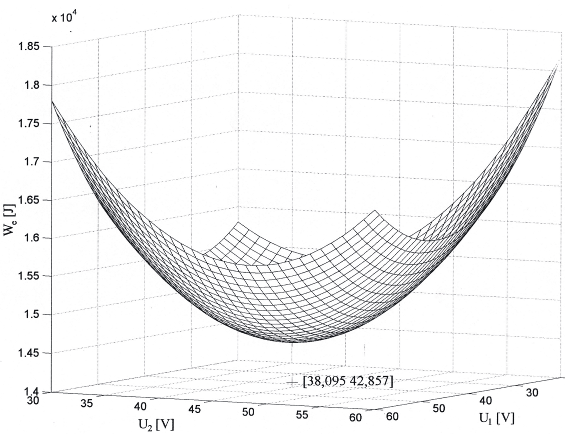

Fig. 2. Energy of electric field as a function of branch voltages U_1 and U_2

Minimum of function $W_e = f(U_1, U_2)$ could be of course defined more easily, in a usual way for seeking the extreme:

$$\frac{\partial W_e}{\partial U_1} = 0 \quad \Rightarrow \quad 6U_1 - 3U_2 = 100$$
$$\frac{\partial W_e}{\partial U_2} = 0 \quad \Rightarrow \quad -3U_1 + 12U_2 = 400 \tag{2}$$

The above mentioned result is found by the solution of these equations. It surely the minimum as we can see from the following inequalities.

$$\frac{\partial^2 W_e}{\partial U_1^2} < 0\,, \qquad \frac{\partial^2 W_e}{\partial U_2^2} < 0$$

We obtain the same result by solving the circuit using some of the known methods, e.g. node-voltage method or Thèvenin´s theorem.

The reader is probably confined by the following facts:

- A general physical principle valid not only in electrical engineering is used for the analysis of the circuits; in electric circuit theory it is not usually used.
- It is interesting that minimising one scalar function, which expresses energy of the circuit, can do the analysis of however complex system.

But the reader might be annoyed that the minimum principle, on which the solution of our example is based, was not sufficiently explained here. We will make it up in the following chapter. We will show that the principle is more general than we can judge from the given example. Also the class of solved problems is much wider than this example shows. Besides electric circuits it is also possible to design electromechanical and magnetic circuits using minimum principle energy. Also non-linear circuits can be considered.

2. KNOWLEDGE FROM ANALYTICAL MECHANICS AND ITS APPLICATION IN ELECTRICAL ENGINEERING

2.1. Double conception of classical mechanics; Hamilton variational principle

Classical mechanics studies the movement of set of material points. It is based on Newton`s laws, in which the basic quantities are force, velocity and momentum.. These are vectors and that is why Newton`s mechanics is sometimes called *vector mechanics*. Applying Newton` s laws on the examined set we obtain movement equations, these differential equations whose integration determines the trajectories of given material points. Newton’s contemporary Gottfried Wilhelm Leibnitz and after him especially Joseph Louis Lagrange showed a different conception: The described the movement of mechanic systems using scalar quantities, e.g. work, kinetic energy etc. Application of this approach led to *analytical mechanics* [1], [3], [8]. Vector mechanics is advantageous with the systems consisting of fewer material points (e.g. examining the movements of planets), but less advantageous with multi-point systems. There is no discrepancy between vector and analytical mechanics; one can be derived from the other. Method difference between both theories is *in the way of the motion equations is defined*, not in the equations.

William Roven Hamilton formulated an important column of analytical mechanics in 1834. It is called *Hamilon´s principle* (*the principle of least action*). It says that *from all possible movements of conservative mechanic system in any time interval such movement occurs for which the functional*

$$S = \int_{t_1}^{t_2} L\, dt \tag{3}$$

reaches extreme (steady) value. The trajectory for which the extreme is realised is called *extremal trajectory (stationar line) of functional S*. Quantity L is called *Lagrange`s function* - shortly *Lagrangian*. For a linear isolated mechanic system the Lagrangian is defined follows:

$$L = T - U \tag{4}$$

where T is kinetic energy and U is potential energy.

Lagrangian L is thus the function of position and velocity. Quantity S is called *action functional* (or shortly *action* or *effect*). For the solved problem we define a corresponding Lagrangian (it is we describe the problem using scalar function) and we require the action functional S to be in its extreme value. From this condition we obtain the equations of the system. Hamilton`s principle can be then described by a mathematical relation

$$\delta S = 0$$

where δS identifies the variation of action functional S.

The mathematical tool for problems of this type is calculus of variations [2], [12]. Formulation of Hamilton’s Principle was modify by various authors (e.g. P. L. Maupertus, C. G. Jacobi, H. Hertz) [10].

Hamilton`s principle provides elegant and brief formulation of dynamical laws. It expresses a general idea, according to which physical phenomena take a course in the simplest and most economical way. Its validity is universal; it is not valid in mechanics only, but also in other areas of physics and thus in electrical engineering.

Lagrange`s analytical mechanics was completed in the first half of the 19th century (Hamilton`s principle was formulated in 1834), it is in time when the theory of electromagnetism was incipient. But even then the idea of using the method of analytical mechanics for mathematical modelling of electric and magnetic phenomena existed. The first one was probably William Thompson (Lord Kelvin), who in 1848 showed that an isolated electrostatic system can be described by potential function, whose course corresponds with the minimum of energy of electrical system - *Thomson`s Principle of Energy*

Minimum. Thompsons`s ideas are further developed by James Clark Maxwell, who (in1873) in his famous work *Treatise on Electricity and Magnetism* [5] applied variational principles on the systems with concentrated parameters, it is on electric circuits. The potential energy in mechanics corresponded with electric energy and kinetic energy of magnetic field. Later Herman Helmholtz (1893) showed the possibility to use Hamilton` s principle in electromagnetic field theory; Maxwell`s equations are also derived from this principle. Nowadays for numerical solution of electromagnetic fields most often the *Method of finite elements* is used, based also on variational principle.

It will be shown now how Hamilton`s principle can be used in the theory of electric circuits and electromechanical systems.

2.2. Generalised coordinates, couplings

Lagrangian is function of time and mechanical or electrical quantities, possibly their first derivations. These quantities are called *generalised coordinates* and labelled $q_1,\dots q_N$. Their first derivations are $q_1',\dots,q_N'$ are called *generalised velocities*. So

$$L = L(q_j, q_j', t), \qquad j = 1,\dots, N$$

Maxwell drew on analogy with analytical mechanics and as generalised coordinates introduced these electrical quantities: charges were generalised coordinates, currents as generalised velocities and magnetic induction fluxes as generalised momentum. In this analogy potential and kinetic energy in mechanics correspond with energy of electric and magnetic field. From geometrical point of view generalised coordinates in N-dimensional *configuration space* describe the examined object.

Dynamic behavior of considered system is limited by certain conditions for generalized coordinates. These conditions are called *constraints*. If they are in form of algebraic equations they are called *holonomic constraints.* If they are expressed in another mathematical form (e.g. inequalities, differential equations etc.), they are called non-holonomic constraints. If the constraints do not depend on time explicitly, they are called *scleronomic*, if time participates in constraint conditions, they are referred to as *rheonomic*. An example of *holonomic* and at the same time *scleronomic* constraint is in the mathematical pendulum: the position of the weight is defined by vector $\boldsymbol{r}$, position of its suspension by vector $\boldsymbol{a}$. The length of the pendulum being l , the following is valid:

$$(\boldsymbol{r} - \boldsymbol{a})^2 = l^2$$

Another example is a point whose position is defined by vector $\boldsymbol{r}$. The point can move in space but must not penetrate in the sphere with radius a, whose center is defined by vector $\boldsymbol{r}_0$. Motion of the point is then defined by non-holonomic scleronomic constraint condition

$$(\boldsymbol{r} - \boldsymbol{r}_0) \geq a^2$$

In electric circuits there can be constraints deduced from Kirchhoff`s laws that express relations in the topological structure of the circuit, e.g. in the incidence between branches and loops. These constraints are characterized by incidence matrix of the circuit. There are mostly holonomic constraints. A system with n constraint conditions is called a system with $s = N - n$ degrees of freedom. From geometrical point of view holonomic constraints can be interpreted as surfaces in space, on which the particles of the system can move. These constraints reduce the N-dimensional configuration space to its s-dimensional subspace.

2.3. Euler-Lagrange equations

In analytical mechanics textbooks (see e.g. [1], [3], [8], [10]) it is proved that for *conservative system with holonomic constraint, in time* $t \in \langle t_1, t_2 \rangle$ *the curve* $q_j(t)$ *satisfy the Hamilton`s principle (i.e. extremal of functional S) only if it is the solution of the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q_j'}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1,2,3 \qquad (5)$$

where s is the number of degrees of freedom.

Equations (5) are called *Euler-Lagrange equations*. They represent a system s ordinary differential equations second order.

For electric circuits these relations are not sufficient, as these circuits are not generally conservative. For non-conservative systems that contain outer forces, sources of energy and elements in which a part of the energy is irreversible transformed into thermal energy (i.e. dissipating elements, in mechanics it is e.g. friction, in electric circuits these are resistors) it is generalized (see e.g. [3], [4], [9], [10], [11]) in the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q'_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial q'_j} = Q_j, \quad j = 1,\dots s \qquad (6)$$

where R is so called *Rayleigh dissipating function*.

By experiment it is possible to prove that with sufficient exactness the following is valid

$$R = \frac{1}{2}\sum_{j=1}^{s} r_j (q'_j)^2 \qquad (7)$$

where: r_j is *parameter of dissipating element* and

Q_j is *generalized force* (i.e. outer force in mechanics or source parameter in circuit).

3. EXAMPLES OF FORMULATION OF EQUATIONS FOR CIRCUIT ANALYSIS

3.1. Capacity circuit

Capacity circuit according Fig.1 was solved in the introduction with reference to "principle of energy minimum". Now let us observe the principle in context of knowledge provided in the previous chapter. The circuit contains only electric field energy, $T = 0$, and Lagrarian $L = -U$. It is evident that Hamilton`s principle is in our case an expression of "minimum energy principle".

Equation (2) can be also obtained from Euler-Lagrange equations. The circuit is conservative and in a steady state, thus equation (5) is valid, in which time derivation is zero. By application of the voltage Kirchhoff`s law on the independent loops the following is obtained

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ 100 \end{bmatrix} = 0$$

Regarding these holonomic scleronomic constraint conditions, the circuit has $s = 2$ degrees of freedom, which means it is possible to describe it by two generalized coordinates. As constraint conditions are known for branch voltages, and as charges $q \approx U$, we can introduce branch voltages as generalized coordinates: $q_1 \approx U_1$, $q_2 \approx U_2$, thus $L = -W_e$, where equation (1) is valid for W_e. Substituting into equation (5), we get equation (2).

3.2. Linear circuit

For linear circuit according Fig. 3 with help of Kirchhoff's current law at independent nodes B_1 and B_2 we obtain holonomic scleronomic constrains conditions in the form

$$\begin{bmatrix} -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = 0$$

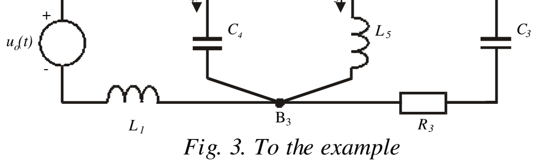

Fig. 3. To the example

Since

$$i_4 = i_1 - i_2 , \qquad i_5 = i_2 - i_3$$

the circuit has $s = 3$ degree of freedom. We use as the generalized coordination the charges q_1 , q_2 , q_3 whence the currents

$$i_1 = \frac{dq_1}{dt}, \quad i_2 = \frac{dq_2}{dt}, \quad i_3 = \frac{dq_3}{dt},$$

Magnetic field energy is

$$T = \frac{1}{2} L_1 q_1'^2 + \frac{1}{2} L_5 (q_2'^2 - q_3'^2)$$

electric field energy is

$$U = \frac{1}{2C_3} q_3^2 + \frac{1}{2C_4} (q_1 - q_2)^2$$

and dissipative function is

$$R = \frac{1}{2} R_1 q_1'^2 + \frac{1}{2} R_2 q_2'^2 + \frac{1}{2} R_3 q_3'^2$$

With substitution this magnitudes into eq. (6) and for Q_j (j = 1, 2, 3) we put $u_0(t)$, 0, 0, we obtain the circuit equation

$$\begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_5 & -L_5 \\ 0 & -L_5 & L_5 \end{bmatrix} \frac{d^2}{dt^2} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} +$$

$$+ \begin{bmatrix} 1/C_4 & -1/C_4 & 0 \\ -1/C_4 & 1/C_4 & 0 \\ 0 & 0 & R_3 + 1/C_3 \end{bmatrix} . \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} u_0(t) \\ 0 \\ 0 \end{bmatrix}$$

The initial condition $q_j(0),\ q_j'(0),\ j = 1,2,3$ we determine from steady state solution of circuit closely before transient state.

3.3. Electromechanical circuit – actuator

In circuit from Fig. 4 is the armature with mass M, the coefficient of damping is B and the coefficient of spring is K. The energy of spring is zero with area gap $x = b$. At the armature act the force $f(t)$ and weight Mg (g is the acceleration of gravity). The resistance of the coil is R_a and inductance is

$$L = \frac{\mu_0 N^2 S}{x}$$

if in magnetic circuit is $\mu \to \infty$ and the magnetic leakage is inconsiderable; S is the cross-section of the armature.

The degree of freedom is $s = 2$ and any constrains conditions. We used following generalised coordinates and generalised forces:

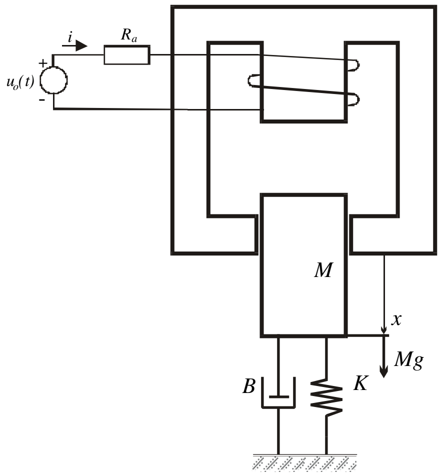

Fig. 4. Electromagnetic actuator

- mechanical (j =1):

$q_1 = x$ … travel of armature ⇒

$q_1' = x' = v$ … velocity of armature

$Q_1 = f(t)$ … drawing force of armature

- electrical (j = 2):

$q_2 = q$ … charge ⇒

$q_2' = q' = i$ … current

$Q_2 = u_0(t)$

We obtain

$$T = \frac{1}{2} M x'^2 + \frac{1}{2} L q'^2$$

$$U = (x - b)^2 / 2K$$

$$F = \frac{1}{2} B x'^2 + \frac{1}{2} R_a q'^2$$

$$Q_1 = M g, \; Q_2 = u_0(t)$$

We compute $L = T - U$ and after substitution into eq. (6) is the equations for transient state:

$$M x'' + B x' + \frac{x-b}{K} + \frac{S q'^2}{x^2} = M g$$

$$\frac{2S}{x} q'' + R_a q - \frac{S q' x'}{x^2} = u_0(t)$$

4. CONCLUSION

When designing a mathematical model for given electrical or electromechanical circuit there are methods based on Kirchhoff`s laws, in electromagnetic circuits also on Newton`s laws, that are analyzed in detail and nowadays commonly used. The submitted article calls attention on an alternative way of deriving motion equations of considered system. It originates in the idea that the circuit is a physical object for which Hamilton`s variational principle is valid. The idea of its application on examination of electromagnetic phenomena can be found in the early stage of the development of this field. Despite the attempts of many writers this approach has not yet been anchored in common electrical engineering practice.

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REFERENCE

[1] Brdička M., Hladík A.: *Theory of mechanics.* (In Czech.) Academia, Praha 1987.

[2] Elsgolc L. E.: *Variation calculus.* (In Czech.) SNTL, Praha 1965.

[3] Gantmacher F.*: Lectures in analytical mechanics.* MIR Publishers, Moskva 1970.

[4] Marx B., Süsse R.: *Theoretische Elektronechnik.* Bd I., II. VDI Verlag, Leipzig, 1994, 1996.

[5] Maxwell J. C.: *A treatise on electricity and magnetism.* Vol.1 and 2, Oxford 1873.

[6] Mayer D.: *Introduction to the electrical circuit theory.* (In Czech.) SNTL/ALFA, Praha, 1st. ed.1978, 2nd ed. 1981.

[7] Mayer D.: *Variations principle and circuit theory.* (In Czech.) Časopis EE, 6 (2000), č. 5. pp. 4-7.

[8] Meirovitch L.: *Methods of analytical dynamics.* McGraw Hill, New York 1970.

[9] Meisel J.*:Principles of electromechanical energy conversion.* McGraw Hill, New York 1966.

[10] Štol I., Tolar J.: *Theoretical physics.* (In Czech.) Edition ČVUT, Praha 1999.

[11] Vodička V.: *Fundamentals of variation calculus for physical and technical applications.* (In Czech.) Elektrotechn. Obzor, 38 (1949), pp. 443-457.

[12] White D. C., Woodson H. W.: *Electromechanical energy conversion.* J. Wiley, London, 1959.