A Simple Encoding Scheme to Achieve the Capacity of Half-Duplex Relay Channel

Zouhair AL-QUDAH[1], *Khalid Ahmad DARABKH*[2]

[1]Communication Engineering Department, Al-Hussein Bin Talal University, 71110 Ma'an, Jordan
[2]Computer Engineering Department, The University of Jordan, Queen Rania St. 11942 Amman, Jordan

qudahz@ahu.edu.jo, k.darabkh@ju.edu.jo

DOI: 10.15598/aeee.v20i1.4347

Article history: Received Aug 30, 2021; Revised Jan 13, 2022; Accepted Feb 16, 2022; Published Mar 31, 2022.

Abstract. *In this paper, the Half-Duplex Relay Channel (HDRC) is thoroughly investigated. Even though this channel model is widely studied, the capacity is not yet fully understood and particularly has not been tightly expressed. In this work, a new capacity expression of the discrete memoryless HDRC is explicitly established. In particular, a new expression of the achievable rate is derived by taking advantage of the well-known capacity results of both the degraded broadcast channel and the multiple access channel. Specifically, in order to obtain the achievable rate, the transmission from the source to destination is operated over two phases. In the first phase, the broadcast phase, the source broadcasts to both relay and destination. In the second phase, both source and relay transmit to destination to form multiple access channel. Then, we prove that the new achievable rate meets the cut-set outer bound such that the capacity of the discrete memoryless HDRC is attained. Next, the new derived capacity result is extended to the case of additive Gaussian channel. Further, the attained capacity is analytically and then numerically shown to encompass all well-known available findings in the literature. Additional numerical examples are also shown to present the cases in which the relay is beneficial and how the achievable capacity varies with the source-relay and relay-destination channel gains.*

Keywords

Broadcast channel, channel capacity, half-fuplex, multiple access channel, relay channel.

1. Introduction

Nowadays, the relay channel has been receiving extensive investigations from both wireless communication and information theoretic perspectives. In relay channel, the transmission from a sender to its destination is supported by at least one intermediate node, the relay [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and [14]. Specifically, in forwarding the sender's message to the intended receiver, the relay may serve either in full-duplex mode or in half-duplex mode [10], [15] and [16]. Despite the fact that the relay channel model with full-duplex operation was extensively studied, the capacity is known in many few different scenarios like the physically degraded relay channel [1], reversely degraded relay channel and semi-deterministic relay channel [17].

On the other hand, the capacity of the HDRC is not yet fully understood. Particularly, many different encoding schemes have been developed for the sake of deriving the capacity of this channel model [4], [6], [5], [7] and [18]. Initially, an outer bound to the achievable rate of transmission over the HDRC was derived in [4]. Then, this outer bound was shown to achieve the capacity in [8]. In this formulation, the transmission from the sender to its destination is performed over two consecutive transmission phases. In addition, the sender's signal is split into two independent parts. In this transmission scheme, particularly the first transmission phase, the source transmits the first part to both the relay and the receiver such that only the relay can decode this part. In the second phase, the source can directly transmit the second part to its destination whereas the relay can forward the first part. By finishing this transmission scheme, the destination can decode the two parts. In another subsequent work, the author in [5] established new bounds regarding the transmission rate over the two-way HDRC. Later, by following the same lines of obtaining the capacity in [8], the authors in [6] designed the polar codes to be suitable for reliable transmission over HDRC. Further, the authors in [7] derived inner and outer transmission bounds over the Gaussian HDRC (GHDRC). Specifically, the achievable rate (inner bound) was characterized in many different cases such that: i) partial decode and forward, or ii) compress and forward, is employed at the half-duplex relay node. Indeed, a cut-set (outer) bound was also developed. Based on their formulation, the derived outer bound and the achievable inner bound are not equal and so the capacity was not attained. In another new recent paper, the authors in [18] investigated and derived the achievable rate of the cooperative Gaussian HDRC. In their formulation, the transmission scheme is split into two transmission phases. In the first phase, the sender employs superposition encoding to transmit to both the relay and receiver. In the second transmission period, only the relay can transmit to forward the its part. In their derivation, one of the main drawbacks is keeping the duration of each phase constant. Another drawback is that only the relay transmits in the second transmission time. Additionally, the authors in [10] proposed a two-phase transmission scheme to develop a new achievable rate of the transmission over Gaussian multiple input multiple output HDRC. In their transmission scheme, specifically in the first phase, the relay partially listens to the source until it can completely decode the source's signal. At that point, the transmission can be successfully completed within the second phase upon the agreement of both the source and relay on transmitting to the receiver.

In this article, we consider the transmission over HDRC. This channel may model a communication scenario in which a user at the cell-edge wants to communicate with a base-station with the help of a half-duplex relay. Another example was introduced in [19], where a Light Emitting Diode (LED) may operate as a half-duplex relay in wireless optical communication. The main goal of this paper is to establish a new achievable capacity of the HDRC with a transmission rate that is primely higher than all known results in the literature. In the way to derive the capacity of the discrete memoryless HDRC, we first derive a new achievable rate. The derivation is based on simple and known encoding schemes. Then, the cut-set upper bound on the capacity is proved to be attained. This new expression of the capacity is also shown to include all known results in the literature.

Particularly, we first derive the achievable rate by employing the well-known encoding and decoding schemes, that are used to achieve the capacity of the degraded Broadcast Channel (BC) [20], [21] and the capacity of the Multiple Access Channel (MAC) [22] and [21]. In the proposed transmission scheme, the source's signal is divided into three independent parts and then transmitted to the receiver over two transmission phases. Particularly, in the first transmission phase, the sender employs the encoding scheme that is usually used by the transmitter of the BC to transmit two independent parts to both the relay node (the stronger receiver) and the receiver (the weaker destination). In the second transmission period, the sender directly transmits the third part whereas the relay forwards its part to the destination. Based on this transmission scheme, a new achievable rate of the transmission over the discrete memoryless HDRC is characterized. This achievable rate is shown to meet the outer bound such that the capacity of the discrete memoryless HDRC is obtained. Then, the achievable capacity is also computed and evaluated in the case of GHDRC. Subsequently, we prove not only theoretically, but also numerically that the derived achievable capacity in this paper has a transmission rate that is undoubtedly higher than all known results in the literature. In particular, two factors are numerically investigated to determine the cases in which the relay is beneficial, i.e. the relay can increase the achievable rate. Specifically, these factors are: i) the channel gain between the source and the relay, and ii) the average power at the relay.

In what follows, specifically in Sec. 2. , the system model is presented. A new achievable rate of the transmission over the discrete memoryless HDRC is derived in Sec. 3. Then, in Sec. 4. , we first show that the derived achievable rate and the outer bound are equal such that the capacity of the discrete memoryless HDRC is attained. Further, the derived capacity is shown to be larger all known results. Next, in Sec. 5. , we extend the capacity of the discrete memoryless HDRC to the Gaussian case. Additionally, some numerical examples are shown in Sec. 6. Finally, the paper is concluded in Sec. 7.

2. System Model

The discrete memoryless HDRC, as depicted in Fig. 1, is composed of: (i) two input alphabets $\mathcal{X}$, and, $\mathcal{X}_{\mathcal{R}}$, and ii) two finite channel output alphabets $\mathcal{Y}_{\mathcal{R}}$, and $\mathcal{Y}_{\mathcal{D}}$. In this channel model, $X \in \mathcal{X}$ and $X_R \in \mathcal{X}_{\mathcal{R}}$, are the input signals generated by the transmitter and the relay, respectively. Additionally, $Y_R \in \mathcal{Y}_{\mathcal{R}}$ and $Y_D \in \mathcal{Y}_{\mathcal{D}}$ represent the output alphabets at the relay and the destination respectively. In this scenario, the source has a message $W \in \{1, ..., 2^{nR}\}$ to be sent to its receiver, with the help of a half-duplex relay, over n channel uses.

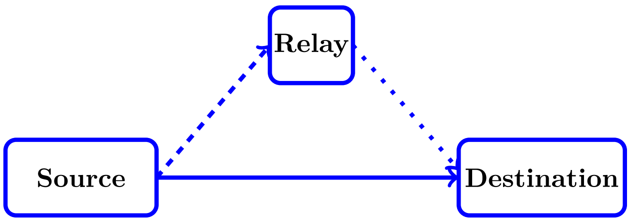

Fig. 1: Half-duplex Relay Channel in which the channels from source to relay and then from relay to destination are orthogonal in time.

Definitely, a rate R is said to be achievable if, for any $\epsilon > 0$, there exists a sequence of (W, n) codes such that the probability of error, P_e^n, goes to 0 for sufficiently large n. In addition, the average probability of error is defined as:

$$P_e^n = \frac{1}{2^{nR}} \sum_{W} P\left[g(Y_D^n) \neq W/W \text{ was sent}\right],$$

where $g(Y_D^n) : \mathcal{Y}_D \longmapsto \hat{W}$ is the decoding function at the destination.

Notations: To easily differentiate between the signals that are sent over the two-phase transmission scheme, a subindex $k \in \{1, 2\}$ is added to the source's signal such that we may have X_k. Likewise, a subindex $k \in \{1, 2\}$ is also added to the received signals such that we may have Y_{D_k}.

3. Achievable Rate of the Discrete Memoryless HDRC

In this section, we derive the achievable rate of the discrete memoryless HDRC. The derivation is based on the well known encoding schemes that are used to achieve the capacity of the degraded BC and MAC. In particular, the transmission from the source to the destination is divided into two phases. In the first phase, which lasts for l uses of the channel such that $\tau = \frac{l}{n}$, the source transmits two different signals to the relay and the destination, as shown in Fig. 2. Then, in the second phase, which lasts for $(n-l)$ uses of the channel such that $\bar{\tau} = \frac{n-l}{n} = 1-\tau$, both the source and the relay transmit their signals to the destination, as shown in Fig. 3. In particular, toward starting transmission, the source splits its message W into three independent parts w_{S_1}, w_R and w_{S_2} so that the total achievable rate is $R = R_1 + R_R + R_2$, where $w_{S_1} \in \{1, \ldots, 2^{nR_1}\}$, $w_R \in \{1, \ldots, 2^{nR_R}\}$, and $w_{S_2} \in \{1, \ldots, 2^{nR_2}\}$. At this point, we are ready to describe the transmission encoding scheme and the decoding process such that the achievable rate of the discrete memoryless HDRC is established.

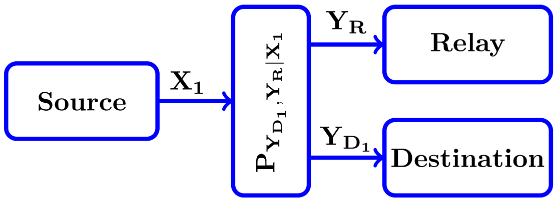

Fig. 2: Transmission over discrete memoryless HDRC in phase 1. The source broadcasts different messages to different receivers.

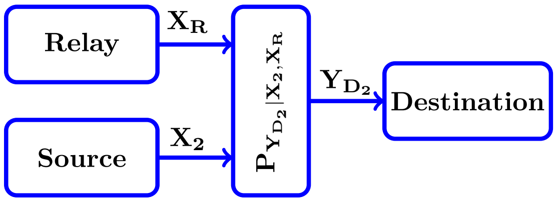

Fig. 3: Transmission over discrete memoryless HDRC in phase 2.

Phase 1: In this phase, the source initially maps the messages w_{S_1} and w_R into $U(w_{S_1})$ and $X_1(w_R, w_{S_1})$, as for encoding at the transmitter of the BC [21]. Specifically, the superposition encoding is employed such that for each codeword $U(w_{S_1})$, independent codewords $X_1(w_R, w_{S_1})$ are generated. Further, the channel from the source to the relay and the destination is assumed to be physically degraded in which $X_1 \rightarrow Y_R \rightarrow Y_{D_1}$ forms a Markov chain. In this case, the relay (the stronger user) can decode both the signals, whereas the destination (the weaker user) can decode only one of the two signals, i.e., $U(w_{S_1})$. To this extent, we are ready to state the following lemma.

Lemma 1. *The achievable rate region of transmitting independent messages from the source to both the relay and destination in the first phase of transmission over the HDRC is the convex-hull of the closure of all rates* (R_1, R_R) *given by:*

$$R_1 \leq \tau I(U; Y_{D_1}) = \psi_1, \tag{1}$$

$$R_R \leq \tau I(X_1; Y_R|U) = \psi_2, \tag{2}$$

for a joint distribution of the form $P(U)P(X_1|U)P(Y_R|X_1)P(Y_{D_1}|Y_R)$. *In this formulation,* $I(.;.)$ *represents the mutual information. It is noteworthy to mention that the destination can improve its estimate regarding the signal* U *at the end of phase 2, as will be shown shortly.*

Proof. The signal generation, encoding and decoding in the first phase follows the same lines that are used for deriving the channel capacity of the degraded BC, as shown in [21]. □

Phase 2: In the second phase, the source and the relay simultaneously encode the messages w_{S_2} and w_R

to generate $X_2(w_{S_2})$ and $X_R(w_R)$, respectively. Consequently, in their transmission to the destination, both the source and the relay form the MAC. In this phase, the achievable rate is given by the following lemma.

Lemma 2. *The achievable rate region of transmitting from both the source and the relay to the destination, in the second phase, is given by the closure of the convex hull of all R_R and R_2 satisfying:*

$$R_R \leq \bar{\tau} I(X_R; Y_{D_2}|X_2), \tag{3}$$

$$R_2 \leq \bar{\tau} I(X_2; Y_{D_2}|X_R) = \psi_3, \tag{4}$$

$$R_2 + R_R \leq \bar{\tau} I(X_R, X_2; Y_{D_2}) = \psi_4, \tag{5}$$

for some product distribution of the form $P_1(X_2)P_2(X_R)$.

Proof. The signal generation, encoding and decoding in the second phase follows the same lines of the proof of deriving the channel capacity of the MAC, as provided in [21]. □

Based on the preceding formulation, we are about to state the achievable rate of the transmission over the HDRC in the following theorem.

Theorem 1. *The achievable rate of the transmission over the discrete memoryless HDRC, C, is given by:*

$$C = \min\{\psi_1 + \psi_2 + \psi_3, \psi_1 + \psi_4\}. \tag{6}$$

Proof. Knowing that the total achievable rate is $R = R_1 + R_R + R_2$ where the sub rates R_1, R_R, and R_2 are previously derived in Eq. (1), Eq. (2), Eq. (3), Eq. (4) and Eq. (5). At this point, the Fourier-Motzkin elimination is applied [23] such that the sub rates R_1, R_R, and R_2 are used to form the total achievable rate, C. □

In summary, in this section, a new achievable rate expression of the transmission over HDRC has been obtained. In the next Section, this new formula is firstly shown to be the capacity of the transmission over HDRC. Then, the achievable capacity is shown to have a transmission rate that is undeniably higher than any previous known result in the literature.

4. Comparisons with Other Results

In this section, we first introduce the available known cut-set outer bound on the transmission over discrete memoryless HDRC. Then, a comparison is made between the cut-set outer bound and the achievable rate from Theorem 1. Based on this comparison, we show that the achievable rate in Theorem 1 is the capacity of the transmission over the discrete memoryless HDRC.

Besides of the above, a comparison is performed with the achievable capacity that was derived in Theorem 4.1.1 [8]. This analysis is performed to make sure that the new established capacity formula has a transmission rate that is certainly higher than the known results.

4.1. Comparison with the Outer-bound

In this subsection, we introduce the available outer bound for transmission over HDRC. Consequently, a comparison is made to show that the achievable rate in Eq. (6) is the channel capacity of the discrete memoryless HDRC.

Lemma 3. *The cut-set (outer-bound) of the transmission over the discrete memoryless HDRC is given by:*

$$R_{out} = \min\{R_{out1}, R_{out2}\},$$

where

$$\begin{aligned} R_{out1} &= \tau I(X_1; Y_R, Y_{D_1}) + \bar{\tau} I(X_2; Y_{D_2}|X_R), \\ R_{out2} &= \tau I(U; Y_{D_1}) + \bar{\tau} I(X_2, X_R; Y_{D_2}). \end{aligned} \tag{7}$$

Proof. The proof can easily be attained from the results of [5] by reducing the two-way half-duplex relay channel into the conventional (one-way) HDRC. Additionally, this outer bound is also derived by the authors in [7]. In particular, a cut-set outer bound on the capacity of the discrete memoryless relay channel is given in Theorem 16.1 [24]. Then, the authors in [7] specialized the outer bound from [24] to the case of HDRC. □

To this end, a comparison is shortly made between the achievable rate that we have derived and the outer bound from the previous lemma. Specifically, we may start with:

$$\begin{aligned} A_1 &= \psi_1 + \psi_2 + \psi_3 - R_{out1} \\ &= \tau I(U; Y_{D_1}) + \tau I(X_1; Y_R|U) + \bar{\tau} I(X_2; Y_{D_2}|X_R) \\ &- [\tau I(X_1; Y_R, Y_{D_1}) + \bar{\tau} I(X_2; Y_{D_2}|X_R)] \\ &= \tau I(U; Y_{D_1}) + \tau I(X_1; Y_R|U) - \tau I(X_1; Y_R, Y_{D_1}) \\ &\overset{(a)}{\geq} \tau I(U; Y_{D_1}) + \tau I(X_1; Y_R|U) \\ &- [\tau I(X_1; Y_{D_1}) + \tau I(X_1; Y_R|U)] \\ &= \tau I(U; Y_{D_1}) - \tau I(X_1; Y_{D_1}) \\ &\overset{(b)}{=} 0, \end{aligned} \tag{8}$$

where the result in step (a) is derived after specializing the mutual information $I(X_1; Y_R, Y_{D_1})$ to the case of physically degraded channel. Additionally, according to [21], this mutual information may be re-written as:

$$I(X_1;Y_R,Y_{D_1}) = I(X_1;Y_{D_1}) + I(X_1;Y_R|U). \quad (9)$$

Further, by letting $U=\emptyset$ and $X_1=\emptyset$, as in [25] and [26], the result in step (b) is obtained.

Additionally, we need to compute:

$$\begin{aligned} A_2 &= \psi_1 + \psi_4 - R_{out2} \\ &= \tau I(U;Y_{D_1}) + \bar{\tau} I(X_R,X_2;Y_{D_2}) \\ &- [\tau I(U;Y_{D_1}) + \bar{\tau} I(X_R,X_2;Y_{D_2})] \\ &= 0. \end{aligned} \quad (10)$$

In conclusion, in light of Eq. (8) and Eq. (10), the derived achievable rate in Theorem 1 is equal to the cut-set outer bound. Therefore, we are now ready to state the following proposition.

Proposition 1. *The achievable rate of the discrete memoryless HDRC, as given in Theorem 1, is the achievable capacity of the discrete memoryless HDRC.*

Proof. The results $A_1 = 0$ and $A_2 = 0$ confirms that the achievable rate that was derived in Theorem 1 meets the outer bound of the discrete memoryless HDRC. Accordingly, the capacity is achieved by the two-phase transmission scheme, as characterized in the previous section. □

4.2. Comparison with the Available Achievable Capacity

In this subsection, a comparison is performed between our result from Proposition 1 and that reported in [8]. In this regard, we first introduce the achievable capacity of the HDRC, as derived in [8]. Then, we show that the achievable capacity that we derive in this paper encompasses the achievable capacity that was derived in [8].

Lemma 4. *The achievable capacity, R_{ach}, of the discrete memoryless HDRC in the case that the relay can decode and then forward the source's signal is given by:*

$$R_{ach} = \min\{R_{ach1}, R_{ach2}\},$$

where:

$$\begin{aligned} R_{ach1} &= \tau I(X_1;Y_R) + \bar{\tau} I(X_2;Y_{D_2}|X_R), \\ R_{ach2} &= \tau I(U;Y_{D_1}) + \bar{\tau} I(X_2,X_R;Y_{D_2}). \end{aligned}$$

Proof. The proof is published in [8]. An outline of the proof is given in the next remark. □

Remark 1. *Toward achieving the capacity in [8], the source's signal is divided into two independent parts. These two parts are sent to the destination, with the aid of the relay node, over two transmission phases. In the first phase, the source transmits the first part to both the relay and the destination. Then, in the second phase, the relay forwards the source's signal to the destination such that the receiver can resolve the uncertainty regarding the part that was already sent by the source in the first phase. Indeed, in this phase, the source can directly transmit the second part to the destination.*

We now compare between the achievable capacity that we derive in this work and the achievable capacity in Lemma 4. In particular, we have:

$$\begin{aligned} A_3 &= \psi_1 + \psi_2 + \psi_3 - R_{ach1} \\ &= \tau I(U;Y_{D_1}) + \tau I(X_1;Y_R|U) + \bar{\tau} I(X_2;Y_{D_2}|X_R) \\ &- [\tau I(X_1;Y_R) + \bar{\tau} I(X_2;Y_{D_2}|X_R)] \\ &= \tau I(U;Y_{D_1}) + \tau I(X_1;Y_R|U) - \tau I(X_1;Y_R) \\ &\overset{(c)}{=} \tau I(U;Y_{D_1}). \end{aligned} \quad (11)$$

We note that the transition to step (c) is made possible by letting $X_1=\emptyset$, as in [25] and [26]. Additionally, the mutual information $\tau I(U;Y_{D_1})$ is always non-negative.

Further, we follow the same lines of obtaining the result in Eq. (10) to confirm that $A_4 = \psi_1 + \psi_4 - R_{ach2}$ is equal to 0.

Proposition 2. *The achievable capacity that we have derived in Proposition 1 has a transmission rate that is higher than the available capacity result in the literature.*

Proof. In light of the values of A_3 and A_4, the derived capacity in this paper has a transmission rate that is higher than the available capacity result in the literature. □

Remark 2. *In [27], G. Kramer, showed that half-duplex channels are a special case of the memoryless full-duplex framework. This is obtained by replacing X_R with the pair (X_R,S_R), where S_R is a binary random variable that indicates the state (i.e., either receiving or transmitting) of the relay. With this formulation, the achievable rate of the physically degraded half-duplex relay channel is given by:*

$$\begin{aligned} R = \sup \min\{&I(X_S,X_R,S_R;Y_D), \\ &I(X_S,Y_R,Y_D|X_R,S_R)\}, \end{aligned} \quad (12)$$

where X_S can be either X_1 or X_2, and Y_D can be either Y_{D_1} or Y_{D_2}. We remark that the authors in [7] showed that the achievable rate in Eq. (12) *is a few bits from the outer-bound of the Gaussian HDRC.*

In this section, we proved that the capacity of the discrete memoryless HDRC, as shown in the Preposition 1, encompasses the available capacity result in the literature. Next, we extend the capacity of the discrete memoryless HDRC into the case of additive Gaussian channel.

5. Capacity of Gaussian HDRC

In this section, we extend the capacity of the HDRC into the Gaussian case. In the first phase, the source employs superposition encoding to form the signal $X_1 = \sqrt{(1-\alpha)}V + \sqrt{\alpha}U$. In this formulation, the signals V and U are independent random variables with zero mean and average power equals to $P_{S_1} = \beta P_1$, where P_1 is the total average transmit power by the source over the two transmission phases. Moreover, α and β are the power allocation factors. Specifically, α, where $0 \le \alpha \le 1$, determines the power allocated to transmit the signals from the source in the first phase. Indeed, β, where $0 \le \beta \le 1$, determines the power allocated to each phase at the source. To finish this stage, the source broadcasts to both the relay and the destination such that the received signals at the relay, Y_R, and the destination, Y_{D_1} are given by:

$$Y_R = h_{sr}X_1 + Z_R, \tag{13}$$

$$Y_{D_1} = h_{sd}X_1 + Z_{D_1}, \tag{14}$$

where h_{sr} and h_{sd} are the channel gains from the source to the relay and the destination, respectively. The noise signals Z_R and Z_{D_1} are independent and identically distributed (i.i.d) Additive White Gaussian Noise (AWGN) signals with zero mean and variances N_R and N_{D_1}, respectively.

In the second phase, both the source and the relay transmit the signals X_2 and X_R, respectively, such that the received signal at the destination, Y_{D_2}, is given by:

$$Y_{D_2} = h_{sd}X_2 + h_{rd}X_R + Z_{D_2}, \tag{15}$$

where h_{rd} is the channel gain from the relay to the destination. Z_{D_2} is the AWGN signal with zero mean and variance N_{D_2}. Additionally, the average transmit power of the signals X_2 and X_R are $P_{S_2} = (1-\beta)P_1$ and P_R, respectively.

After this construction, we are about to compute the sub-rates that appear in the capacity of the discrete memoryless HDRC in (6). For example:

$$\begin{aligned}\psi_1^* &= \tau I(U; Y_{D_1}) \\ &= \tau h(Y_{D_1}) - \tau h(Y_{D_1}|U) \\ &= \tau C\left(\frac{|h_{sd}|^2 \alpha P_{S_1}}{N_{D_1} + |h_{sd}|^2(1-\alpha)P_{S_1}}\right),\end{aligned} \tag{16}$$

where $h(X)$ is the differential entropy of the random variable X, and $C(x) = \log(1+x)$. The rest of the sub-rates are similarly computed as follows:

$$\begin{aligned}\psi_2^* &= \tau C\left(\frac{|h_{sr}|^2(1-\alpha)P_{S_1}}{N_R}\right), \\ \psi_3^* &= \bar{\tau} C\left(\frac{|h_{sd}|^2 P_{S_2}}{N_{D_2}}\right), \\ \psi_4^* &= \bar{\tau} C\left(\frac{|h_{sd}|^2 P_{S_2} + |h_{rd}|^2 P_R}{N_{D_2}}\right).\end{aligned} \tag{17}$$

After this formulation, we are eager to introduce the capacity of the GHDRC.

Theorem 2. *The achievable capacity of the Gaussian HDRC,C^*, is given by:*

$$C^* = \max_{\alpha,\beta} \min\{\psi_1^* + \psi_2^* + \psi_3^*, \psi_1^* + \psi_4^*\}, \tag{18}$$

where the sub-rates ψ_1^*, ψ_2^*, ψ_3^*, and ψ_4^* are given in Eq. (16) *and Eq.* (17), *respectively.*

Proof. The achievable capacity of the GHDRC can be obtained by plugging the generated codewords and the received signals to the achievable capacity of Preposition 1. For instance, the derivation of ψ_1^* is given in Eq. (16). □

Remark 3. *We note that the destination can improve its estimate regarding the signal U, which was sent in the first phase. In particular, at the end of phase 2, the destination can decode the relay's signal, X_R, and then use this estimation to generate the signal X_1. After that, the destination can remove the effect of X_1 from Y_{D_1}. In conclusion, we may redefine the achievable sub-rate of ψ_1 as follows:*

$$\begin{aligned}\grave{\psi}_1 &= \tau I(U; Y_{D_1}|X_1) \\ &= \tau C\left(\frac{|h_{sd}|^2 \alpha P_{S_1}}{N_{D_1}}\right).\end{aligned} \tag{19}$$

However, in this case, the achievable rate is larger than the available cut-set outer-bound. Hence, a new outer-bound has to be derived.

Remark 4. *We note that the achievable capacity in Theorem 2 can be maximized by optimizing: i) the power allocation factors α and β, and ii) the duration of the BC phase and the MAC phase. In particular, the achievable capacity can be maximized by making $\psi_1^* + \psi_2^* + \psi_3^* = \psi_1^* + \psi_4^*$.*

Remark 5. *To find the optimal value of τ, we may need to solve the equality, as appeared in the previous remark, with respect to τ. In particular, the optimal duration of the first phase is given by $\tau = \frac{A}{B}$, where:*

$$\begin{aligned}A &= C\left(\frac{|h_{sd}|^2 P_{S_2} + |h_{rd}|^2 P_R}{N_{D_2}}\right) - C\left(\frac{|h_{sd}|^2 P_{S_2}}{N_{D_2}}\right), \\ B &= C\left(\frac{|h_{sr}|^2(1-\alpha)P_{S_1}}{N_R}\right) + A.\end{aligned} \tag{20}$$

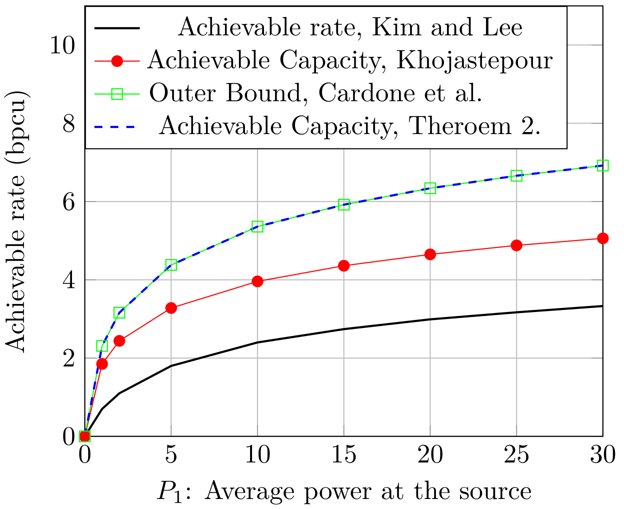

Fig. 4: Comparison between the achievable capacity in Theorem 2 and the known available results in the literature.

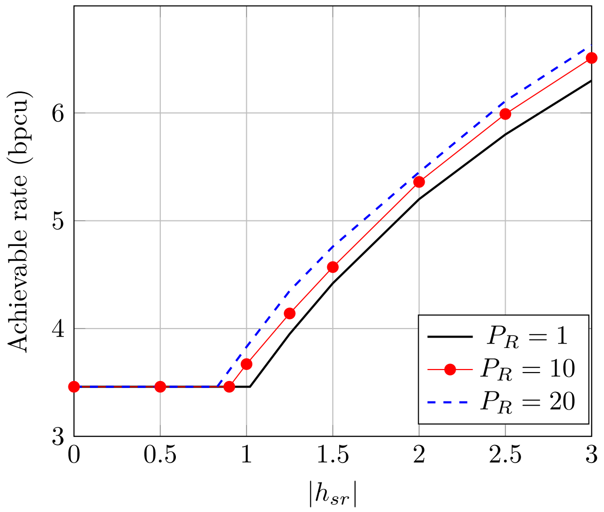

Fig. 5: The relation between the achievable rate and the channel gain, $|h_{sr}|$, for different values of the average power at the relay, P_R.

6. Numerical Results

In this section, we numerically compare between the available results and the derived achievable capacity of the GHDRC in Theorem 2. In this numerical evaluation, the average transmit power at both source and relay is set to 10. Unless otherwise mentioned, the channel gains $|h_{sd}|$, $|h_{sr}|$, and $|h_{rd}|$ are set to 1, 2, and 2, respectively. Moreover, the noise variances N_{D_1}, N_{D_2}, and N_R are normalized to 1.

Figure 4 compares the results presented in [8], [5], [7] and [18] and the derived capacity in Theorem 2. First, we present the achievable rate which was derived by Kim and Lee in [18]. Second, we show the achievable capacity which was derived by Khojastepour in [8]. Third, as established by both Stein in [5] and Cardone et al. in [7], an outer bound to the achievable rate is also drawn. Fourth, we also show the achievable capacity which we derive in Theorem 2. It can be clearly seen that the achievable capacity in Theorem 2: i) has a transmission rate higher than the available known results in [8], and ii) meets the outer bound of the HDRC. We remark that this numerical result agrees fairly with the analytical result, as shown in Sec. 4.

After showing that the achievable capacity that we have derived encompasses all available results in the literature, we investigate two numerical examples to address the effect of relay location. Particularly, as soon as the relay moves toward or outward the source (destination), the channel gains between the relay and the end nodes (source and destination) change.

In this example, we want to check the cases in which the relay is beneficial. Specifically, the achievable rate (capacity), without employing the relay, is given by $\log_2(1 + |h_{1D}|^2 P_1) = \log_2(11) = 3.46$. Figure 5 shows that both the source-relay channel gain, $|h_{sr}|$, and the average power at the relay, P_R, play a main role in determining whether to use the relay in transmission or not. In particular, for low values of both: i) the average power, i.e. $P_R < 1$, and ii) the channel gain, i.e., $|h_{sr}| < 1$, the relay cannot increase the achievable capacity. Thus, the source needs to directly transmit its signal to the destination with a constant rate, which is equal to 3.46, as mentioned before. In the case of strong channel gain, i.e. $|h_{sr}| > 1$, the relay can significantly increase the achievable capacity. Further, as the average transmit power P_R increases so does the achievable rate.

Now, we show the relation between the optimum duration of the broadcast phase, phase 1, and the channel gain $|h_{sr}|$ for different values of the channel gain $|h_{rd}|$. In specific, Fig. 6 shows that as the channel gain $|h_{sr}|$ gets stronger, i.e. the relay is moving toward the source, the duration of the broadcast phase reduces. In this case, the relay can easily get the source's signal. Further, the relay is given more time to cooperate and then forward the source's signal into its destination. This figure also presents that the percentage of the first phase is inversely proportional with the channel gain $|h_{rd}|$.

Finally, as depicted in Fig. 7, we numerically show that the achievable capacity has a concave relation with the duration of the first phase, τ. In specific, the value of τ, at which the achievable capacity gets its maximum, varies with the channel gains $|h_{sr}|$ and $|h_{rd}|$. For example, in the case of $|h_{sr}| = 3, |h_{rd}| = 1$, the relay can easily decode the source's signal, so, the duration of the first phase is short. On the other hand, in the case of $|h_{sr}| = 1, |h_{rd}| = 3$, the relay requires more channel uses before decoding the source's signal.

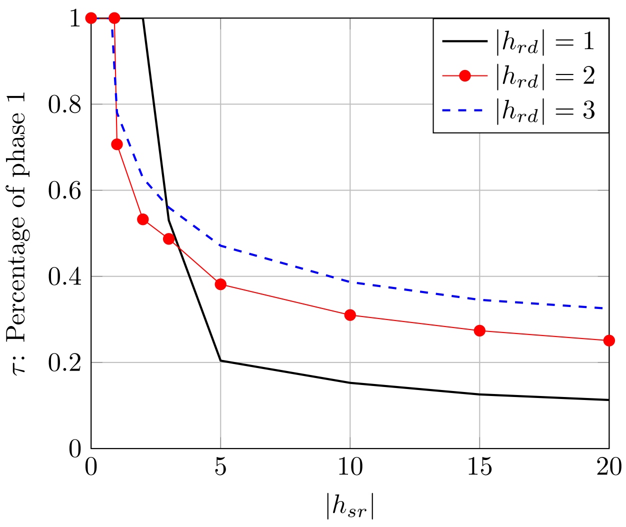

Fig. 6: The relation between percentage of the first phase and the channel gain, $|h_{sr}|$, for different values of the channel gain, $|h_{rd}|$.

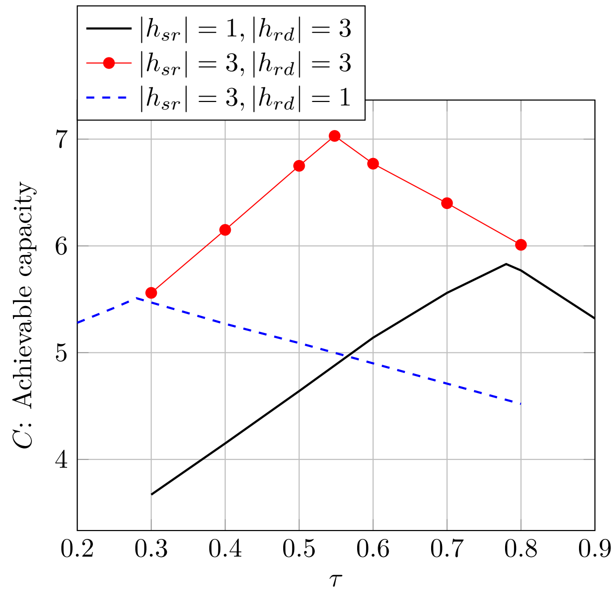

Fig. 7: The relation between the achievable capacity and τ for different values of the channel gain, $|h_{sr}|$ and $|h_{rd}|$.

Therefore, the duration of the first phase is longer than the previous case. Finally, in the case that the relay has strong channel gains with both source and relay, i.e. $|h_{sr}| = 3, |h_{rd}| = 3$, the relay can quickly obtain the source's signal. Then, in the second phase, the relay forwards the source's signal into its destination. In this case, due to strong channel gain $|h_{rd}| = 3$, the forwarded signal from the relay plays a main role in increasing the achievable capacity.

7. Conclusions and Future Works

In this paper, the capacity of the discrete memoryless HDRC is established by using the well-known encoding and decoding schemes that are normally used to achieve the capacity of both the degraded BC and the MAC. In the way to develop the capacity, an achievable rate of the discrete memoryless HDRC is firstly derived. Then, this achievable rate is shown to achieve the cut-set upper bound such that the capacity is determined. Thereafter, the capacity of the discrete memoryless HDRC is also extended to the additive Gaussian case. The major finding of this work is that the analytical and numerical results of aforementioned capacities agree fairly with each other. Indeed, the established capacity that we have derived has a transmission rate that is certainly higher than all well-known available rates in the literature. Future work may include: (i) extending the capacity result to the two-way relay channel in which the relay operates in half-duplex mode, and (ii) designing a new practical encoding and decoding schemes to achieve the capacity.

Author Contributions

Z.A. developed both the theoretical and numerical analyses. K.A.D. contributed to the final version of the manuscript.

References

[1] COVER, T. and A.E. GAMAL. Capacity theorems for the relay channel. *IEEE Transactions on Information Theory*. 1979, vol. 25, iss. 5, pp. 572–584. ISSN 1557-9654. DOI: 10.1109/TIT.1979.1056084.

[2] EL GAMAL, A. and S. ZAHEDI. Capacity of a class of relay channels with orthogonal components. *IEEE Transactions on Information Theory*. 2005, vol. 51, iss. 5, pp. 1815–1817. ISSN 1557-9654. DOI: 10.1109/TIT.2005.846438.

[3] AL-QUDAH, Z. Achievable rates of Gaussian two-way relay channel with orthogonal components and partial decoding at relay. *Electronics Letters*. 2016, vol. 52, iss. 3, pp. 200–202. ISSN 0013-5194. DOI: 10.1049/el.2015.3598.

[4] KHOJASTEPOUR, M. A., A. SABHARWAL and B. AAZHANG. Bounds on Achievable Rates for General Multi-terminal Networks with Practical Constraints. In: *Information Processing in Sensor Networks*. Heidelberg: Springer, 2003, pp. 146–161. ISBN 978-3-540-02111-7. DOI: 10.1007/3-540-36978-3_10.

[5] STEIN, M. *Two-way Relay Channels with Half-duplex Constraint*. 2010, Munchen. Thesis. Technische Universitat Munchen. Supervisor: Prof. Dr. Ing. Wolfgang Utschick.

[6] DUO, B., Z. WANG, X. GU and Q. GUO. Achieving the capacity of half-duplex degraded relay channels using polar coding. *Chinese Journal of Aeronautics*. 2014, vol. 27, iss. 3, pp. 584–592. ISSN 1000-9361. DOI: 10.1016/j.cja.2014.04.008.

[7] CARDONE, M., D. TUNINETTI, R. KNOPP and U. SALIM. On the Gaussian Half-Duplex Relay Channel. *IEEE Transactions on Information Theory*. 2014, vol. 60, iss. 5, pp. 2542–2562. ISSN 1557-9654. DOI: 10.1109/TIT.2014.2309614.

[8] KHOJASTEPOUR, M. A. Distributed Cooperative Communications in Wireless Networks. 2005, Houston. Thesis. Rice University. Advisor Dr. Behnaam Aazhang.

[9] AL-QUDAH, Z. Achievable rates of a state-dependent relay channel with orthogonal components. *IET Communications*. 2016, vol. 10, iss. 16, pp. 2149–2153. ISSN 1751-8636. DOI: 10.1049/iet-com.2015.0798.

[10] AL-QUDAH, Z., A. ALQUDAH and K. A. DARABKH. Transmission over Gaussian MIMO half-duplex relay channel. *Physical Communication*. 2020, vol. 40, iss. 1, pp. 1–7. ISSN 1874-4907. DOI: 10.1016/j.phycom.2020.101089.

[11] AL-QUDAH, Z. and A. ABABNEH. Broadcast diamond channel: Transmission strategies and bounds. *Journal of the Franklin Institute*. 2021, vol. 358, iss. 3, pp. 2178–2193. ISSN 0016-0032. DOI: 10.1016/j.jfranklin.2020.12.016.

[12] AL-SAWALMEH, W., Z. AL-QUDAH and K. A. DARABKH. Multiple access relay channel: Achievable rates over orthogonal channels. *AEU - International Journal of Electronics and Communications*. 2019, vol. 109, iss. 1, pp. 121–127. ISSN 1434-8411. DOI: 10.1016/j.aeue.2019.06.036.

[13] AL-JAAFREH, M. and Z. AL-QUDAH. Transmission rates over state-dependent diamond channel. *IET Communications*. 2018, vol. 12, iss. 4, pp. 409–414. ISSN 1751-8636. DOI: 10.1049/iet-com.2017.0517.

[14] AL-QUDAH, Z. and A. MUSA. On the capacity of the state-dependent interference relay channel. *International Journal of Communication Systems*. 2019, vol. 32, iss. 14, pp. 1–16. ISSN 1099-1131. DOI: 10.1002/dac.4079.

[15] AL-QUDAH, Z., M. AL BATAINEH and A. MUSA. A novel multiple access diamond channel model. *International Journal of Communication Systems*. 2020, vol. 33, iss. 17, pp. 1–12. ISSN 1099-1131. DOI: 10.1002/dac.4616.

[16] AL-QUDAH, Z., M. H. S. ALRASHDAN and K. A. DARABKH. On the capacity region of the multiple access half-duplex relay channel. *International Journal of Communication Systems*. 2021, vol. 34, iss. 17, pp. 1–14. ISSN 1099-1131. DOI: 10.1002/dac.4990.

[17] GAMAL, A. E. and M. AREF. The capacity of the semideterministic relay channel. *IEEE Transactions on Information Theory*. 1982, vol. 28, iss. 3, pp. 536–536. ISSN 1557-9654. DOI: 10.1109/TIT.1982.1056502.

[18] KIM, J.-B. and I.-H. LEE. Capacity Analysis of Cooperative Relaying Systems Using Non-Orthogonal Multiple Access. *IEEE Communications Letters*. 2015, vol. 19, iss. 11, pp. 1949–1952. ISSN 1558-2558. DOI: 10.1109/LCOMM.2015.2472414.

[19] KIM, S.-M. and H.-J. LEE. Half-duplex visible light communication using an LED as both a transmitter and a receiver. *International Journal of Communication Systems*. 2016, vol. 29, iss. 12, pp. 1889–1895. ISSN 1099-1131. DOI: 10.1002/dac.2921.

[20] COVER, T. M. Comments on broadcast channels. *IEEE Transactions on Information Theory*. 1998, vol. 44, iss. 6, pp. 2524–2530. ISSN 1557-9654. DOI: 10.1109/18.720547.

[21] COVER, T. M. and J. A. THOMAS. *Elements of information theory*. 2nd ed. Hoboken: Wiley-Interscience, 2006. ISBN 978-0-471-24195-9.

[22] ABU AL HAIJA, A. and M. VU. Rate Maximization for Half-Duplex Multiple Access with Cooperating Transmitters. *IEEE Transactions on Communications*. 2013, vol. 61, iss. 9, pp. 3620–3634. ISSN 1558-0857. DOI: 10.1109/TCOMM.2013.071013.120173.

[23] SCHRIJVER, A. *Theory of Linear and Integer Programming*. 1st ed. Amsterdam: Wiley, 1998. ISBN 978-0-471-98232-6.

[24] GAMAL, A. E. and Y.-H. KIM. *Network Information Theory*. 1st ed. New York: Cambridge University Press, 2012. ISBN 978-1-107-00873-1.

[25] RINI, S., D. TUNINETTI, N. DEVROYE and A. J. GOLDSMITH. On the Capacity of the Interference Channel With a Cognitive Relay. *IEEE Transactions on Information Theory*. 2014,

vol. 60, iss. 4, pp. 2148–2179. ISSN 1557-9654. DOI: 10.1109/TIT.2014.2301454.

[26] AL-QUDAH, Z. and D. RAJAN. An Achievable Region for the Cognitive Interference Relay Channel. *IEEE Transactions on Cognitive Communications and Networking*. 2018, vol. 4, iss. 4, pp. 665–676. ISSN 2332-7731. DOI: 10.1109/TCCN.2018.2863382.

[27] KRAMER, G. Models and theory for relay channels with receive constraints. In: *42nd Annual Allerton Conference on Communication, Control, AND Computing*. Monticello: University of Illinois, 2004, pp. 1312–1321. ISBN 978-1-604-23660-6.

About Authors

Zouhair AL-QUDAH (corresponding author) received the B.Sc., M.Sc., and Ph.D. all in Electrical Engineering from Yarmouk University, Jordan, in Sep. 2002, Kalmar University College, Sweden in Nov. 2006, and Southern Methodist University at Dallas, Texas in May 2013, respectively. Since August 2013, he has been with Al-Hussein Bin Talal University at Ma'an, Jordan, where he is currently an Associate Professor. His research interest includes coding and information theory in wireless communication systems, cooperativecommunications 5G communication, resource allocation problems, interference management and signal processing.

Khalid Ahmad DARABKH received the Ph.D. degree in Computer Engineering from the University of Alabama in Huntsville, USA, in 2007 with honors. He has joined the Computer Engineering Department at the University of Jordan as an Assistant Professor since 2007 and has been a Tenured Full Professor since 2016. He is engaged in research mainly on wireless sensor networks, queuing systems and networks, multimedia transmission, and steganography and watermarking. He authored and co-authored of at least a hundred research articles and served as a reviewer in many scientific journals and international conferences. Prof. Darabkh is the recipient of 2016 Ali Mango Distinguished Researcher Reward for Scientific Colleges and Research Centers in Jordan. He serves on the Editorial Board of Telecommunication Systems, published by Springer, Computer Applications in Engineering Education, published by John Wiley & Sons, and Journal of High Speed Networks, published by IOS Press. Additionally, he serves as a TPC member of many reputable Institute of Electrical and Electronics Engineers (IEEE) conferences such as Global Communications Conference (GLOBECOM), Local Computer Networks (LCN), Vehicular Technology Conference (VTC-Fall), Personal, Indoor and Mobile Radio Communications (PIMRC), International Symposium on Wireless Communication Systems (ISWCS), and International Association of Environmental Analytical Chemistry (IAEAC). Moreover, he is a member of many professional and honorary societies, including Eta Kappa Nu, Tau Beta Pi, Phi Kappa Phi, and Sigma XI. He was selected for inclusion in the Who's Who Among Students in American Universities and Colleges and Marquis Who's Who in the World. As administrative experience at the University of Jordan, he served as Assistant Dean for Computer Affairs in the College of Engineering from Sept 2008 to Sept 2010. Additionally, he served as Acting Head of the Computer Engineering Department from June 2010 to Sept 2012.