D–POSETS WITH MEET FUNCTION

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Summary For any two observables on a full tribe we can always construct a two–dimensional observable. In this case the crucial role is played by pointwise multiplication of the functions of tribe. Riečan and Mundici, ([6]) studied semisimple MV–algebras enriched by an additional product operation. The abstract product operation on MV–algebra is relating with the natural pointwise multiplication of functions. The sequential product on effect algebras defined by Gudder and Geechie, ([3]) is based on the similar idea as above. In this paper the meet function as a generalization of the product is defined and the properties of D–posets with meet function are presented.

1. BASIC NOTIONS

D-posets ([5]), or effect algebras ([2]), are joint generalizations of the classical algebraic structures, which are algebraic models of the quantum mechanics and the fuzzy sets theory systems. In these systems the important roll play the MV–algebras ([1]), which are a non–commutative generalization of the Boolean algebras. Examples of σ–complete MV–algebras are given by tribes, a generalization of σ–algebras of sets.
A state and an observable are the basic notions of the probability theory on MV–algebras.

Definition 1. ([5]) The structure ($\mathcal{P}$, ≤, ™, 1) is called a D–poset if *b* ™ *a* is defined iff $a \leq b$ and

(D1) *b* ™ *a* ≤ *b*.

(D2) *b* ™ (*b* ™ *a*) = *a*.

(D3) If $a \leq b \leq c$, then *c* ™ *b* ≤ *c* ™ *a* and (*c* ™ *a*) ™ (*c* ™ *b*) = *b* ™ *a*.

The partial sum operation on D–poset is defined by the following formula

a © *b* = 1™ ((1™ *a*) ™ *b*)

for *a, b* such that *b* ≤ 1 ™ *a*.
D-poset which is a lattice is called D-lattice. In this case the difference operation is totally defined and
b – *a* = *b* ™ (*a* T *b*).

Definition 2. ([4]) The structure ($\mathcal{P}$, ≤, –, 0, 1) is called a boolean D–poset (MV–algebra) if $b - a$ is defined for every *a,b* 5 $\mathcal{P}$ and the following conditions are satis–fied:

(BD1) $a - 0 = a$ for every *a* 5 $\mathcal{P}$.

(BD2) *a, b* 5 $\mathcal{P}$, $a \leq b$ implies $c - b \leq c - a$ for every *c* 5 $\mathcal{P}$.

(BD3) $b - (b - a) = a - (a - b)$ for every *a,b* 5 $\mathcal{P}$.

(BD4) $(c - b) - a = (c - a) - b$ for every *a,b,c* 5 $\mathcal{P}$.

The lattice operation, meet of elements *a,b* from $\mathcal{P}$ is defined as

a T *b* = $b - (b - a) = a - (a - b)$.

Definition 3. ([6]) A product on MV–algebra $\mathcal{M}$ is a commutative and associative binary operation on $\mathcal{M}$ satisfying the following conditions, for all *a,b,c* 5 $\mathcal{M}$.

(P1) 1 Φ *a* = *a*.

(P2) *c* Φ (*b* – *a*) = *c* Φ *b* – *c* Φ *a*.

The product on MV–algebra $\mathcal{M}$ has the following properties.

Proposition 1. ([6]) Let $\mathcal{M}$ be an MV–algebra. Then

a) 0 Φ *a* = 0 for all *a* 5 $\mathcal{M}$.

b) If $a \leq b$ then *a* Φ *c* ≤ *b* Φ *c* for all *c* 5 $\mathcal{M}$.

c) *a* – *a* Φ *c* ≤ 1 – *b*.

d) *a* – (1 – *b*) ≤ *a* Φ *b* ≤ *a* T *b*.

e) *If a* – (1 – *b*) = *0* then *c Φ* (*a* © *b*) = *c* Φ *a* © *c* Φ *b*.

Definition 4. ([3]) A commutative sequential product on an effect algebra $\mathcal{E}$ is a commutative binary operation on $\mathcal{E}$ satisfying the following conditions, for all *a, b, c*∈ $\mathcal{E}$:

(GG1) 1 Φ *a* = *a*.

(GG2) If $a \perp b$ then *c* Φ *a* ⊥ *c* Φ *b* and

c Φ (*a* © *b*) = *c* Φ *a* © *c* Φ *b*.

The property (GG2) we can write by the operation difference as follows:

If $a \leq b$ then *c* Φ *a* ≤ *c* Φ *b* and *c* Φ (*b* – *a*) = *c* Φ *b* – *c* Φ *a*.

The product *a* Φ *b* of elements *a, b* from MV–algebra $\mathcal{P}$ provides the element *d*, *d* ≤ *a, b* such that *a* – *d* ≤ 1 – *b*. About such element *d* says too the notion of the compatibility of elements *a, b* from a D–poset.

Definition 5. ([4]) Two elements *a, b* of a D-poset are compatible, if there exists an element *d* 5 $\mathcal{P}$, *d* ≤ *a, d* ≤ *b,* such that *a* ™ *d* ≤ 1 ™ *b*.

2. MEET FUNCTION ON D–POSETS

The element d in the previous definition is not unique in general. In the classical case the element d is a unique and $d = a$ T b.
In an MV-algebra every two elements a, b are compatible and

(comp) $a - (1 - b) \leq d \leq a \wedge b.$

The product $a \, \Phi \, b$ of elements a, b of an MV-algebra is one element of the elements d fulfilling the definition 5.

Exapmle 1. Let an MV-algebra is a unit interval [0, 1] with the difference $y - x = \max\{y - x, 0\}$, $y - x$ is a difference of reals. Then

$$\max\{y + x - 1, 0\} \leq d \leq \min\{x, y\}.$$

By the comparison of the properties d) and (comp) we may to admit the idea of a generalization of the product on MV–algebras and on D-posets in the following way. We will to define a function $\delta: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, such that the product of elements a, b is one example of the function δ.

Definition 6. Let ($\mathcal{P}, \leq$, 0, 1, ™) be a D-poset. A **meet function** on $\mathcal{P}$ is a map $\delta: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ fulfilled the following conditions for every $a, b, c \in \mathcal{P}$.

(J1) $\delta(a, b) = \delta(b, a)$

(J2) $\delta(\delta(a, b), c) = \delta(c, \delta(b, c))$.

(J3) $\delta(1, a) = a$.

(J4) If $a \leq b$, then $\delta(a, c) \leq \delta(b, c)$.

(J5) a ™ $\delta(a, b) \leq 1$ ™ b.

Example 2. The product on an MV-algebra $\mathcal{M}$ is a meet function on $\mathcal{M}$.

Example 3. The commutative sequential product on an effect algebra $\mathcal{A}$ is a meet function on $\mathcal{A}$.

Example 4. Let $\mathcal{S}$ be a σ-algebra of subsets of a non-empty set $\mathcal{Z}$. Let χ_A, ($\chi_A \equiv$ A) be a characteristic function of a set A $\subseteq \mathcal{Z}$. The map $\delta: \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $\delta(A, B) = \chi_A \cdot \chi_B$ is a meet function on $\mathcal{S}$.

Example 5. Let $\mathcal{L}$ be a D–lattice. Then the following two maps $\delta: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$

$$\delta_{max}(a, b) = b - (b - a) = a - (a - b) = a \wedge b$$

$$\delta_{min}(a, b) = b - (1 - a) = a - (1 - b) = a \, . \, b$$

are meet functions on $\mathcal{L}$.

The meet function has the following properties.

Proposition 2. Let a, b are the elements of D-poset $\mathcal{P}$. Let $\delta: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ be a meet function on $\mathcal{P}$. Then

(i) $\delta(a, b) \leq a, b$.

(ii) $\delta(a, 0) = 0$.

Proof. The assertion is a straightforward corollary of the properties (J3), (J4). □

Proposition 3. Let $\delta: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ be a meet function on D-lattice $\mathcal{P}$. Then

$$\delta_{min}(a, b) = a. \, b \leq \delta(a, b) \leq a \text{ T } b = \delta_{max}(a, b)$$

for every $a, b \in \mathcal{P}$.

Proof. We recall that in this case the difference operation is total and $b - a = b$ ™ (a T b). From the property (J5) of the meet function and from the properties of the difference we have

$$a - (1 - b) \leq a - (a - \delta(a, b)) = \delta(a, b).$$

From the property (i) $\delta(a, b) \leq a \wedge b$. □

Theorem 1. For the meet functions δ_{max} and δ_{min} on a D–lattice $\mathcal{L}$ the following inequalities hold for every $a, b, c \in \mathcal{L}$, $a \leq b$

(i) $(b - a) \, . \, c \leq (b \, . \, c) - (a \, . \, c)$,

(ii) $(b - a) \wedge c \geq (b \wedge c) - (a \wedge c)$.

Proof. Let $a, b, c \in \mathcal{L}$, $a \leq b$.

(i) Then $b - a \leq b - (a - (1 - c))$ and so

$$\begin{aligned}
(b - a) \, . \, c &= (b - a) - (1 - c) \\
&= \big((b - a) - ((1 - c) \wedge b)\big) \vee ((b - a) - b) \\
&= (b - a) - ((1 - c) \wedge b) \\
&\leq \big(b - (a - (1 - c))\big) - ((1 - c) \wedge b) \\
&= \big(b - ((1 - c) \wedge b)\big) - (a - (1 - c)) \\
&= (b \, . \, c) - (a \, . \, c)
\end{aligned}$$

(ii) For assumtions we have $b \geq b \wedge c$ and

$$\begin{aligned}
b - a &\geq (b \wedge c) - a \\
&= (b \wedge c) - (b \wedge c \wedge a) \\
&= (b \wedge c) - (c \wedge a).
\end{aligned}$$

Then $(b - a) \wedge c \geq ((b \wedge c) - (c \wedge a)) \wedge c$

$$= (b \wedge c) - (c \wedge a),$$

because $((b \wedge c) - (c \wedge a)) \leq c$. □

Theorem 2. Let $\delta: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ be a meet function on D–poset $\mathcal{P}$. Then the function $\sigma: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$,

$\sigma(a, b) = 1$ ™ $((1$ ™ $b)$ ™ $(a$ ™ $\delta(a, b)))$

fulfils the following conditions:

(i) $\sigma(a, b) = 1$ ™ $((1$ ™ $a)$ ™ $(b$ ™ $\delta(a, b)))$

(ii) $\sigma(a, b) \geq a$, $\sigma(a, b) \geq b$.

(iii) $\sigma(a, b)$ ™ $a = b$ ™ $\delta(a, b)$, or

$\sigma(a, b)$ ™ $b = a$ ™ $\delta(a, b)$

respectively.

Proof. To prove the assertion (i) we start from the expression $(1 \text{™} b) \text{™} (a \text{™} \delta (a, b))$. Then

$$(1 \text{™} b) \text{™} (a \text{™} \delta (a, b))$$

$$= (1 \text{™} b) \text{™} ((1 \text{™} \delta (a, b)) \text{™} (1 \text{™} a))$$

$$= (1 \text{™} b) \text{™} \big[((1 \text{™} \delta (a, b)) \text{™} (b \text{™} \delta (a, b))) \text{™} (1 \text{™} a) \text{™} (b \text{™} \delta (a, b)))\big]$$

$$= (1 \text{™} b) \text{™} \big[(1 \text{™} b) \text{™} ((1 \text{™} a) \text{™} (b \text{™} \delta (a, b)))\big]$$

$$= (1 \text{™} a) \text{™} (b \text{™} \delta (a, b)).$$

So the equality

$$1 \text{™} ((1 \text{™} b) \text{™} (a \text{™} \delta (a, b)))$$

$$= 1 \text{™} ((1 \text{™} a) \text{™} (b \text{™} \delta (a, b)))$$

is true.

(ii) Because $(1 \text{™} a) \text{™} (b \text{™} \delta (a, b)) \leq 1 \text{™} a$ we have

$a = 1 \text{™} (1 \text{™} a) \leq 1 \text{™} ((1 \text{™} a) \text{™} (b \text{™} \delta (a, b))) = \sigma(a, b)$.

Similarly $\sigma (a, b) \geq b$.

(iii) We calculate

$$\sigma (a, b) \text{™} a = \big[1 \text{™} ((1 \text{™} a) \text{™} (b \text{™} \delta (a, b)))\big] \text{™} a$$

$$= (1 \text{™} a)((1 \text{™} a) \text{™} b \text{™} \delta (a, b)))$$

$$= (b \text{™} \delta (a, b)).$$

Similarly $\sigma (a, b) \text{™} b = a \text{™} \delta (a, b)$. □

Definition 7. The function $\sigma (a, b)$ on a D–poset is called a *joint function*.

3. CONCLUSION

The D–posets or MV–algebras are aximatic models of the non–Kolmogorovian (non–commutative) probability theory. The basic notions in this case are the state (probability measure) and the observable (random variable).

The meet function on D–posets defined in this article is a generalization of the product on MV–algebras, so it will to play a crucial role in the construction of the two–dimensional observable.

The assertions of Theorem 2 are very interesting for D–posets with states and observables on its. For examle, the assertion (iii) of this theorem implies the valuation of the state.

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