Global Optimization Using Space Filling Curves

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Abstract. *The existence of space filling curves opens the way to reducing multivariate optimization problems to the minimization of univariate functions. In this paper, we analyze the Hoelder continuity of space filling curves and exploit this property in the solution of global optimization problems. Subsequently, an algorithm for minimizing univariate Hoelder continuous functions is presented and analyzed. It is shown that the algorithm computes the approximate minimum with the guaranteed precision. The algorithm is tested on some types of two-dimensional functions.*

Keywords

Dimension reduction, global optimization, Hoelder continuity, Lipschitz continuity, space filling curves.

1. Introduction

Many engineering problems lead to a multivariate global optimization. Such issue can be difficult to solve. An approach presented in this paper is to turn the problem into its one-dimensional equivalent. A way how to provide such simplification is to use a space filling curve.

This paper deals with the problem of searching a global minimum

$$F(y^*) = \min_{y \in D} F(y), \tag{1}$$

and a global minimizer $y^* \in D$, where D is an N-dimensional hypercube defined as follows

$$D = \{y \in \mathbb{R}^N, -\frac{1}{2} \leq y_j \leq \frac{1}{2}, 1 \leq j \leq N\}. \tag{2}$$

The objective function F is assumed to satisfy the Lipschitz condition with a constant L, $0 < L < \infty$. The main idea of the paper is to turn the multivariate optimization problem into its one-dimensional equivalent, which can be solved using some techniques evolved for univariate optimization. One way how to do so is to develop a continuous correspondence y mapping a one-dimensional interval onto the hypercube. The problem Eq. (1) turns into the following one

$$F(y^*) = F(y(x^*)) = \min_{x \in [0,1]} F(y(x)), \tag{3}$$

where $x^* \in [0, 1]$. A complete analysis is done to prove that a univariate algorithm based on Hoelder continuity can be used to find an approximation of the optimal value of F over the domain D. The performance of the algorithm is illustrated by numerical experiments.

2. Space Filling Curves

Main ideas in this section are inspired by [1] and [3].

Definition 1. *A space filling curve is a single-valued continuous correspondence y mapping the unit interval $[0, 1]$ onto the hypercube D from Eq. (2).*

If y is a space filling curve, then

$$F(y^*) = \min_{x \in [0,1]} F(y(x)). \tag{4}$$

Though the concept of a space filling curve is useful for the analysis of the algorithm, the effective computation is usually based on a continuous correspondence y_n mapping the unit interval only into D. The following theorem gives some information about the optimal value of $F \circ y_n$ using the quality of y_n.

Theorem 1. *Let (y_n), where $y_n : [0, 1] \to D$, be a sequence of curves such that*

$$\sup_{y \in D} \operatorname{dist}(y_n([0, 1]), y) =: \varepsilon_n \to 0, \tag{5}$$

for $n \to \infty$ *and let* (x_n^*) *be an arbitrary sequence in* $[0,1]$ *satisfying*

$$F(\underbrace{y_n(x_n^*)}_{=:y_n^*}) = \min_{x\in\,[0,1]} F(y_n(x)). \tag{6}$$

Then:

1. $(\forall n \in\, \mathbb{N}) : 0 \le F(y_n^*) - \min_{y\in\,D} F(y) \le L\varepsilon_n,$ (7)

2. *if* $\widetilde{y} \in\, D$ *is an accumulation point of* (y_n^*)*, then* $F(\widetilde{y}) = \min_{y\in\,D} F(y)$.

Proof.
1. Let us assume that

$$F(y^*) := \min_{y\in\,D} F(y) \le F(y_n^*) \le F(y_n(x_n)), \tag{8}$$

where $x_n \in\, [0,1]$ is chosen so that

$$\|y_n(x_n) - y^*\| \le \varepsilon_n.$$

Hence

$$\begin{aligned} 0 \le F(y_n^*) - F(y^*) \le F(y_n(x_n)) - F(y^*) \le \\ \le L(\|y_n(x_n) - y^*\|) \le L\varepsilon_n. \end{aligned} \tag{9}$$

The last inequality is based on the Lipschitz continuity of F on D and on the quality of y_n. This completes the proof of Eq. (7).
2. If $\widetilde{y}$ is an accumulation point of (y_n^*), then there exists a subsequence (for the sake of clarity labeled the same way as the original sequence) such that

$$y_n^* \to \widetilde{y}. \tag{10}$$

Using the continuity of F leads to

$$F(y_n^*) \to F(\widetilde{y}), \tag{11}$$

but, from Eq. (9), we also get

$$F(y_n^*) \to F(y^*). \tag{12}$$

It follows that

$$F(\widetilde{y}) = F(y^*) = \min_{y\in\,D} F(y). \tag{13}$$

□

Remark 1. *If* $y^* = \arg\min_{y\in\,D} F(y)$ *is unique, then*

$$y_n^* \to y^*. \tag{14}$$

Using y_n reduces the multidimensional optimization problem into its one-dimensional equivalent that can be solved by univariate algorithms. If such method provides a lower bound M_n of the one-dimensional function $F \circ y_n$, then M_n is obviously a lower bound of F along y_n. Is it possible to establish a lower bound for F over the whole D? The following theorem gives an answer.

Theorem 2. *Assume that the curve* y_n*,* $n \in\, \mathbb{N}$*, satisfies the assumptions of Thm. 1 and*

$$(\forall x \in\, [0,1]) : M_n \le F(y_n(x)). \tag{15}$$

Then the value

$$M = M_n - L\varepsilon_n \tag{16}$$

is a lower bound of F *over the entire* D*, i.e.*

$$M \le \min_{y\in\,D} F(y). \tag{17}$$

Proof. Using the result and notation of the previous theorem, we get

$$\begin{aligned} (\forall y \in\, D): \ F(y) \ge F(y^*) = (F(y^*)- \\ -F(y_n(x_n))) + F(y_n(x_n)) \ge -L\varepsilon_n + M_n. \end{aligned} \tag{18}$$

□

In what follows, we assume that $F \circ y_n$ is Hoelder continuous with some real constants $H \ge 0, \alpha \in\, (0,1\rangle$, i.e.

$$\begin{aligned} (\forall x', x'' \in\, [0,1]) : \\ |F(y_n(x')) - F(y_n(x''))| \le H|x' - x''|^\alpha. \end{aligned} \tag{19}$$

Let us describe such curve for N-dimensional case. At first, divide the domain D into 2^N equal hypercubes. Number all of the subcubes using the index z_1, $0 \le z_1 \le 2^N - 1$. Each subcube with the index z_1 is designated $D(z_1)$. Moreover,

$$D = \bigcup_{i=0}^{2^N-1} D(z_i). \tag{20}$$

Using the same approach, divide each of the subcubes from the previous partitioning into 2^N equal subcubes and number them with the index z_2, $0 \le z_2 \le 2^N - 1$. Each subcube of the second partitioning is now designated $D(z_1, z_2)$. Continuing the same process we get hypercubes $D(z_1, z_2, \ldots, z_M)$ and the edge length will be 2^{-M}. The total number of the subcubes in M-th partition will be equal to 2^{MN} and

$$\begin{aligned} D \supset D(z_1) \supset D(z_1, z_2) \supset \ldots \\ \cdots \supset D(z_1, z_2, \ldots, z_M). \end{aligned} \tag{21}$$

Now cut the interval $[0,1]$ into 2^N equal subintervals. Every single part is designated $d(z_1)$, $0 \le z_1 \le 2^N - 1$. In the same manner, cut all the subintervals once again, etc. Continuing the same process we get 2^{MN} subintervals with the length equal to 2^{-MN}, which are designated $d(z_1, z_2, \ldots, z_M)$. Moreover,

$$\begin{aligned} [0,1] \supset d(z_1) \supset d(z_1, z_2) \supset \ldots \\ \cdots \supset d(z_1, z_2, \ldots, z_M). \end{aligned} \tag{22}$$

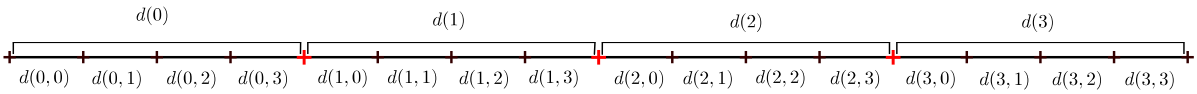

Fig. 1: The first and the second partition of the unit interval.

The process is illustrated by Fig. 1. The interval $d(z_1, z_2, \ldots, z_M)$ can be also written as

$$d(z_1, z_2, \ldots, z_M) = [x, x + 2^{-MN}], \tag{23}$$

and can be referred to as $d(M, x)$. The corresponding subcube $D(z_1, z_2, \ldots, z_M)$ is designated $D(M, x)$. The process of partitioning has to satisfy the following condition.

Condition 1. *If the subintervals $d(M, v')$ and $d(M, v'')$, $M \in \mathbb{N}$, have a common end point, then the corresponding subcubes $D(M, v')$ and $D(M, v'')$ have a common face.*

Now consider a space filling curve $y : [0,1] \overset{\text{onto}}{\to} D$ such that

$$(\forall M \in \mathbb{N}_0) : y(d(z_1, \ldots, z_M)) \subset D(z_1, \ldots, z_M). \tag{24}$$

Then $F \circ y$ is Hoelder continuous with the constants $2L\sqrt{N+3}$ and $\frac{1}{N}$, i. e. for all $x', x'' \in [0,1]$

$$|F(y(x')) - F(y(x''))| \leq 2L\sqrt{N+3}(|x' - x''|)^{\frac{1}{N}}. \tag{25}$$

The proof can be found in [1]. In what follows, we use Hilbert-type curves. The process of partitioning for two-dimensional Hilbert-type curve is illustrated by the following figure.

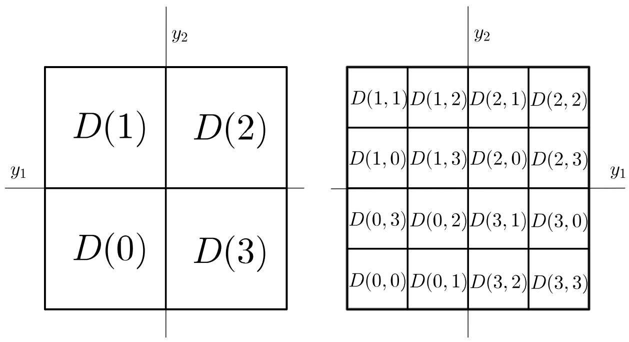

Fig. 2: The first and the second partition of the cube D for two-dimensional Hilbert-type curve.

For the practical computation, we use only "iterates" of the Hilbert curve. Consider the result of the M-th partition illustrated in this section. The construction of the "M-th iteration" of the Hilbert curve y_M is described for example in [1] and [3]. The curve

$$y_M : [0,1] \overset{into}{\to} D, \tag{26}$$

satisfies the following properties. Let $0 = x_0 < x_1 < \cdots < x_{2^{MN}} = 1$ be the end points of the subintervals $d(z_1, \ldots, z_M)$ and let

$$y_i := y_M(x_i), \qquad i \in \{0, \ldots, 2^{MN}\}. \tag{27}$$

Then

$$y_M(x) = y_i + (y_{i+1} - y_i)2^{MN}(x - x_i), \tag{28}$$

$$\|y_{i+1} - y_i\| \leq 2^{-M}, \tag{29}$$

where $x \in [x_i, x_{i+1}], i \in \{0, \ldots, 2^{MN} - 1\}$, and

$$(\forall K < M) : x_i \in d(z_1, \ldots, z_K) \Rightarrow y_i \in D(z_1, \ldots, z_K). \tag{30}$$

The curves y_M for $M = 0, 1, 2$ and $N = 2$ are illustrated by Fig. 3. The construction of the curves

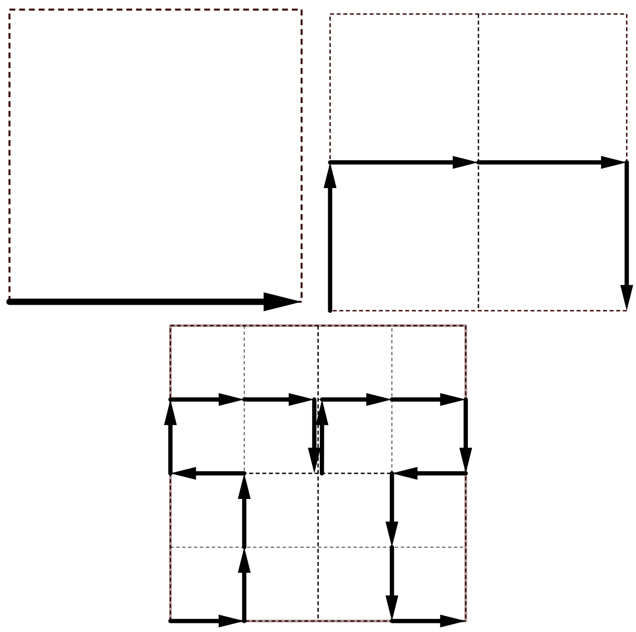

Fig. 3: Iterations of the Hilbert curve for $N = 2$.

is based on complex transformations and is completely described in [3]. The next goal is to prove that also $F \circ y_M$ satisfies the Hoelder condition similar to Eq. (25).

Theorem 3. *The function $F \circ y_M$, $M \in \mathbb{N}$, fulfills the Hoelder condition with the constants $2L\sqrt{N+3}$ and $\frac{1}{N}$ on interval $[0,1]$, i.e. for all x', $x'' \in [0,1]$*

$$\begin{aligned} &|F(y_M(x')) - F(y_M(x''))| \leq \\ &\leq 2L\sqrt{N+3}(|x' - x''|)^{\frac{1}{N}}. \end{aligned} \tag{31}$$

Proof. Let x', $x'' \in [0,1]$, $x' \neq x''$. Then there exists $n \in \mathbb{N}_0$ such that

$$2^{-(n+1)N} \leq |x' - x''| \leq 2^{-nN}. \tag{32}$$

The proof can be divided into two parts.
1. Suppose that $n \geq M$. If x' and x'' are from the same subinterval $d(M, x_i)$, then, according to the properties of y_M, the points $y_M(x')$ and $y_M(x'')$ belong to the same linear segment with end points y_i, y_{i+1}. From Eq. (28), Eq. (29) and Eq. (32), we get

$$\begin{aligned} &\|y_M(x') - y_M(x'')\| \leq \|y_i + (y_{i+1} - y_i)2^{MN} \cdot \\ &\cdot (x' - x_i) - y_i - (y_{i+1} - y_i)2^{MN}(x'' - x_i)\| = \\ &= 2^{MN}\|y_{i+1} - y_i\||x' - x''| \leq 2^{MN}2^{-M}2^{-nN}. \end{aligned} \tag{33}$$

Using the first inequality in Eq. (32), it follows that

$$\begin{aligned} &\|y_M(x') - y_M(x'')\| \leq \\ &\leq 2^{(M-n)(N-1)}2|x' - x''|^{1/N} \leq 2|x' - x''|^{1/N}. \end{aligned} \tag{34}$$

Hence

$$\begin{aligned} &|F(y_M(x')) - F(y_M(x''))| \leq \\ &L\|y_M(x') - y_M(x'')\| \leq 2L|x' - x''|^{\frac{1}{N}}. \end{aligned} \tag{35}$$

If $x' \in d(M, x_i)$ and $x'' \in d(M, x_{i+1})$, then the points $y_M(x')$, $y_M(x'')$ belong to two different segments with a common end point y_{i+1}. Using the previous result, we get

$$\begin{aligned} &\|y_M(x') - y_M(x'')\| \leq \|y_M(x') - y_{i+1}\| + \\ &+ \|y_M(x'') - y_{i+1}\| \leq 4|x' - x''|^{1/N}. \end{aligned} \tag{36}$$

Hence

$$|F(y_M(x')) - F(y_M(x''))| \leq 4L|x' - x''|^{\frac{1}{N}}. \tag{37}$$

2. Now suppose that $n \in \mathbb{N}_0$, $n < M$. If x', x'' are both from $d(n, \tilde{x}_i)$, then, from Eq. (30), $y_M(x')$, $y_M(x'') \in D(n, \tilde{x}_i)$. The maximal distance can be estimated as follows

$$\|y_M(x') - y_M(x'')\| \leq 2^{-n}\sqrt{N}. \tag{38}$$

If $x' \in d(n, \tilde{x}_i)$, $x'' \in d(n, \tilde{x}_{i+1})$, then $y_M(x') \in D(n, \tilde{x}_i)$ and $y_M(x'') \in D(n, \tilde{x}_{i+1})$. According to the Cond. 1, $D(n, \tilde{x}_i)$ and $D(n, \tilde{x}_{i+1})$ are contiguous. Therefore

$$\|y_M(x') - y_M(x'')\| \leq 2^{-n}\sqrt{N+3}. \tag{39}$$

Using Eq. (32), we can derive the final estimate

$$\|y_M(x') - y_M(x'')\| \leq 2\sqrt{N+3}|x' - x''|^{1/N}. \tag{40}$$

Since the function F is Lipschitz continuous, it follows that

$$\begin{aligned} &|F(y_M(x')) - F(y_M(x''))| \leq \\ &\leq L\|y_M(x') - y_M(x'')\| \leq \\ &\leq 2L\sqrt{N+3}|x' - x''|^{1/N}, \end{aligned} \tag{41}$$

which completes the proof. □

Remark 2. *Let M_M be the lower bound of $F \circ y_M$ and let $N = 2$. Then*

$$(\forall y \in D) : M_M - L2^{-M} \leq F(y). \tag{42}$$

3. Optimization Algorithm

The algorithm itroduced in this section is inspired by [1], [2] and is designed for the optimization of the univariate functions that are Hoelder continuous. Consider $g : [a,b] \to \mathbb{R}$ satisfying the Hoelder condition with the constants $H, \alpha, a, b \in \mathbb{R}$, $a < b$. Let $x_0, x_1, \ldots, x_n$ be the trial points obtained in the previous iterations. These points divide $[a,b]$ into n intervals. Let I_i be an interval with end points x_j, x_k that are consecutive, $j, k \in \{0, \ldots, n\}$, $x_j < x_k$. Then the lower bounding function on the interval I_i is constructed as follows:

$$l_i^n(x) := \max\{l_i^L(x), l_i^R(x)\}, \tag{43}$$

where

$$l_i^L(x) = g(x_j) - H(x - x_j)^\alpha, \tag{44}$$

$$l_i^R(x) = g(x_k) - H(x_k - x)^\alpha. \tag{45}$$

The functions l_i^L and l_i^R are illustrated by Fig. 4. It can be shown that the function

$$L^n(x) := l_i^n(x), \qquad x \in I_i, i \in \{0, \ldots, n-1\} \tag{46}$$

is a lower bounding of g over $[a,b]$.

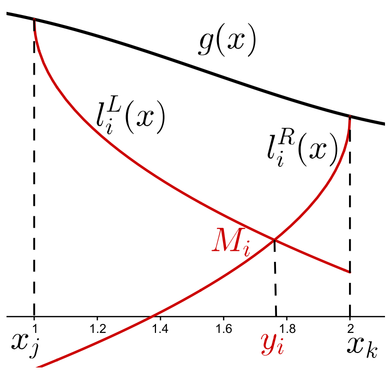

Fig. 4: The functions l_i^L and l_i^R.

The algorithm computes a value $y_i \in I_i$ as an x coordinate of the intersection of l_i^L and l_i^R. Then, the characteristic M_i is computed in the following way

$$M_i = l_i^L(y_i) = l_i^R(y_i). \tag{47}$$

It is true that $M_i \leq \inf\{g(x), x \in I_i\}$. In Fig. 4, we can see a graphical interpretation of y_i and M_i.

The interval I_t with the lowest value of M_t is chosen and $x_{n+1} := y_t$ becomes the next trial point, that divides I_t into two subintervals. For both of them, new characteristics are computed. The algorithm goes on until

$$\mathrm{diam}(I_t) \geq \delta. \tag{48}$$

Let us denote the approximation of the minimizer generated by the algorithm by $\bar{x}$. Let ε be the desired precision of the algorithm, i.e. the goal is to find $\bar{x} \in [a, b]$ such that

$$g(\bar{x}) - \min_{x\in [a,b]} g(x) \leq \varepsilon. \tag{49}$$

The value $\delta = \delta_\varepsilon$ in Eq. (48) can be chosen so that

$$\delta \leq 2\left(\frac{\varepsilon}{H}\right)^{\frac{1}{\alpha}}. \tag{50}$$

The steps of the algorithm can be described as follows:

- **first iteration:** Set $x_0 = a$ and $x_1 = b$ and compute the values $g(x_0)$ and $g(x_1)$. Then the functions l_0^L and l_0^R are constructed, similarly to the Fig. 4. The point y_0 is found as the x coordinate of their intersection and the characteristic M_0 is computed using Eq. (47). If $M_0 = g(y_0)$, the optimization is done and $\bar{x} = y_0$. Otherwise, the algorithm sets the next trial point $x_2 = y_0$, that divides I_0 into two subintervals, $I_0 = [x_0, x_2]$ and $I_1 = [x_2, x_1]$. For both intervals, the values y_0, y_1 and the corresponding characteristics M_0, M_1 are computed.
- **n-th iteration** Let $x_0, x_1, \ldots, x_n$ be the trial points gained from the previous iterations, not necessarily sorted. Let $I_0, I_1, \ldots, I_{n-1}$ be the intervals (generated in the previous steps) with the end points x_i, $i \in \{0, \ldots, n\}$. These intervals are characterized by the values $y_0, y_1, \ldots, y_{n-1}$ and $M_0, M_1, \ldots, M_{n-1}$ computed in a way described above. The algorithm chooses the interval I_t such that

$$M_t = \min\{M_0, M_1, \ldots, M_{n-1}\}, \tag{51}$$

and sets $x_{n+1} = y_t$. If $M_t = g(y_t)$, then $\bar{x} = y_t$ and the optimization is done. Otherwise, the point y_t divides I_t into two subintervals, I_t and I_n. For both subintervals, the points y_t, y_n and the characteristics M_t, M_n are computed. If

$$\mathrm{diam}(I_t) \leq \delta, \tag{52}$$

the algorithm computes the approximation of the optimal value

$$g^* = \min\ \{g(x_i), 0 \leq i \leq n + 1\}, \tag{53}$$

and the minimizer

$$\bar{x} = \arg\min\ \{g(x_i), 0 \leq i \leq n + 1\}. \tag{54}$$

Otherwise, the next iteration is done in the same manner.

For illustration, the first 6 iterations for the function

$$g(x) = (x - 0.3)^2 + 1, \tag{55}$$

on the interval $[0, 1]$ were chosen. The red lines illustrate the curves l_i^L and l_i^R and the optimal values of M_i are marked by the red dots.

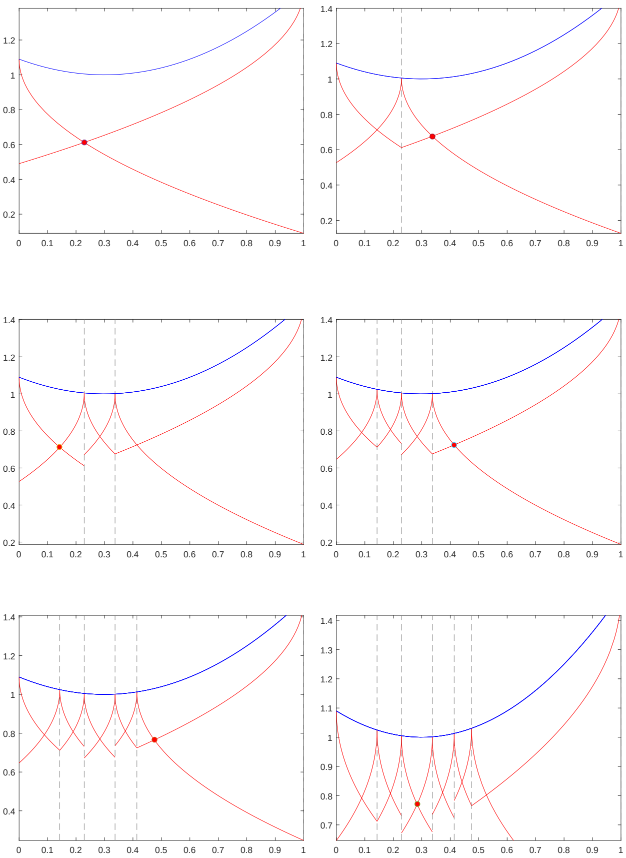

Fig. 5: First 6 iterations for the function from Eq. (55).

The algorithm can be used to compute the optimal value of F along the curve y_M. As we have shown in the previous section, the function $F \circ y_M$ is Hoelder continuous on $[0, 1]$ with the constants $\alpha = \frac{1}{N}$ and $H = 2L\sqrt{N+3}$. In the following observations, we assume that $N = 2$. We get

$$\sup_{y\in D} \mathrm{dist}(y_M([0, 1]), y) = \frac{1}{2^M}. \tag{56}$$

Let us return to the problem Eq. (1). If $\varepsilon > 0$ is a desired precision, i.e. if we want to find $\bar{y} \in D$ such that

$$F(\bar{y}) - \min_{y \in D} F(y) \leq \varepsilon, \tag{57}$$

then we can choose the parameters M and δ so that

$$M \geq \log_2 \frac{2L}{\varepsilon}, \tag{58}$$

$$\delta \leq 2\left(\frac{\varepsilon}{2H}\right)^2. \tag{59}$$

To prove the statement, we use Thm. 1, the quality of y_M, the inequality Eq. (50) and a triangular inequality and we get

$$|F(\bar{y}) - \min_{y \in D} F(y)| \leq |F(\bar{y}) - \min_{x \in [0,1]} F(y_M(x))| + \\ + |\min_{x \in [0,1]} F(y_M(x)) - \min_{y \in D} F(y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{60}$$

The algorithm described above is rather theoretical. For the practical computation, some simplifications have to be made.

For the two-dimensional case it is not difficult to compute the intersection of l_i^L and l_i^R, but for the high-dimensional problem it is more useful to approximate l_i^L and l_i^R by two lines, as we can see in Fig. 6. The value $\tilde{y}_i$ can be computed as the first

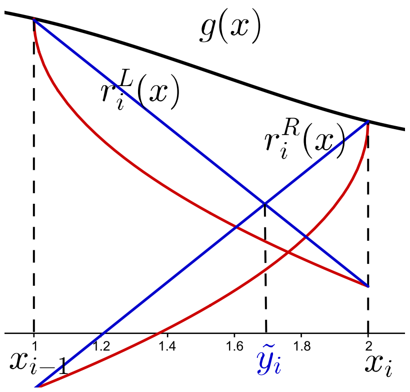

Fig. 6: The approximations of l_i^L and l_i^R.

coordinate of their intersection and

$$\tilde{M}_i = \min\{l_i^L(\tilde{y}_i), l_i^R(\tilde{y}_i)\}. \tag{61}$$

It is also more convenient to use an approximated value of the Hoelder constant. In [2], two possible approximations are proposed.

The choice of the interval I_t can be improved. Two methods are presented in [2].

4. Numerical Experiments

For our experiments, three types of functions were chosen. Let us start with the function

$$F_1(x, y) = (x - 0.3)^2 + (y - 0.7)^2 + 1. \tag{62}$$

It is not difficult to realize that $y^* = [0.3, 0.7]$ and $F(y^*) = 1$. The algorithm was tested for $M = 1, 2, \ldots, 10$ and the precision 10^{-3}. The results are summarized in the Tab. 1.

Tab. 1: Results summarization.

M	$F(y_M(\bar{x})) - F(y^*)$	$y_M(\bar{x})$	$\|y_M(\bar{x}) - y^*\|$
1	$4 \cdot 10^{-2}$	[0.2998,0.5000]	0.2000
2	$2.5 \cdot 10^{-3}$	[0.3009,0.7500]	0.0500
3	$5.1 \cdot 10^{-3}$	[0.2500,0.7518]	0.0720
4	$1.5 \cdot 10^{-4}$	[0.3125,0.7016]	0.0126
5	$1.5 \cdot 10^{-4}$	[0.3016,0.6875]	0.0126
6	$7.6 \cdot 10^{-5}$	[0.3082,0.7031]	0.0087
7	$1.1 \cdot 10^{-4}$	[0.2891,0.6994]	0.0110
8	$5.6 \cdot 10^{-5}$	[0.2930,0.7026]	0.0075
9	$2.9 \cdot 10^{-5}$	[0.2949,0.7019]	0.0054
10	$1.8 \cdot 10^{-5}$	[0.2959,0.7013]	0.0043

In Fig. 7, we can see the solution for $M = 5$.

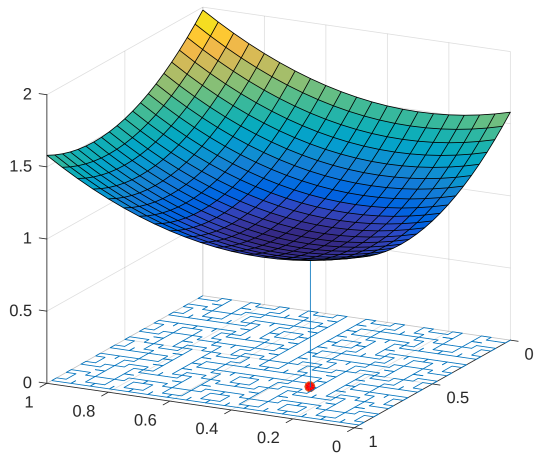

Fig. 7: The function F_1 together with the fifth level of the Hilbert curve and the approximate minimizer.

Consider next the multiextremal function

$$F_2(x, y) = -0.5 \sin(2\pi x) \sin(2\pi y) + 1. \tag{63}$$

The algorithm found one of the minimizers and the result is illustrated by Fig. 8.

The last tested function is

$$F_3(x, y) = \sqrt{(x - 0.6)^2 + (y - 0.4)^2} + 0.5. \tag{64}$$

As we can be seen Fig. 9, the algorithm works also for the nonsmooth function.

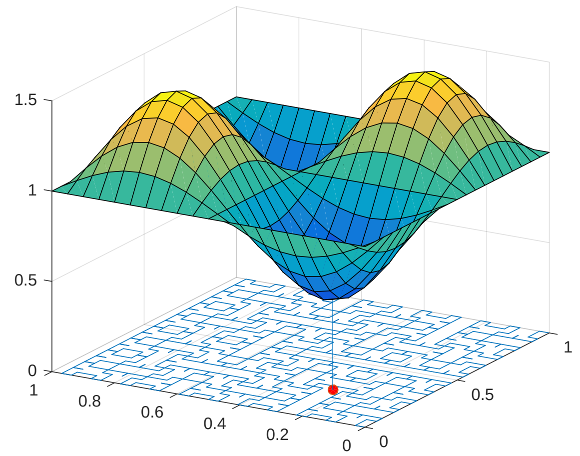

Fig. 8: The function F_2 together with the fifth level of the Hilbert curve and the approximate minimizer.

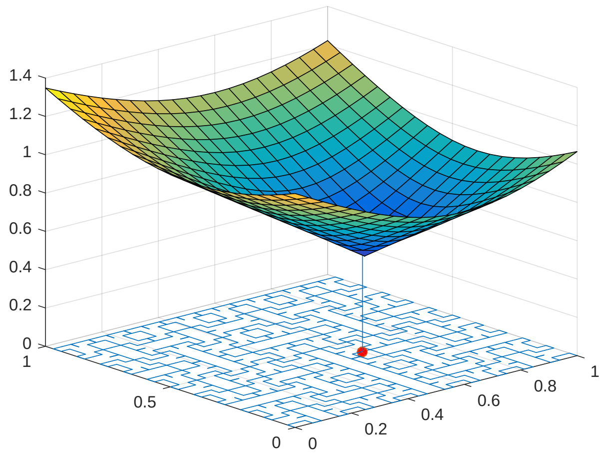

Fig. 9: The function F_3 together with the fifth level of the Hilbert curve and the approximate minimizer.

5. Conclusion

In this paper, we described a minimization algorithm based on reducing the problem by means of Hilbert-type space filling curve. Subsequently, complete mathematical analysis was done to show that the function $F \circ y_M$ is Hoelder continuous. Thus an algorithm for minimizing univariate Hoelder continuous functions could be used to find an approximation of the optimal value of F. It was shown that the algorithm computed the approximate minimum of F with the guaranteed precision. The algorithm and the analysis will be extended for the dimensions $N > 2$ and used for some practical problems of the multivariate optimizations.

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