Coupling of Finite and Boundary Elements for Transient Eddy Current Problems

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Abstract. *A symmetric coupling of methods of finite and boundary elements for numerical solution of transient eddy current problems is described. This is an essential step in modelling of electromagnetic forming of metalic sheets. The finite element method is employed in the conducting region of the metalic sheet. The boundary element method relies on the Stratton-Chu representation formula and it models the electromagnetic field in the air including its decay at infinity. We impose external currents by the Biot-Savart law.*

Keywords

Boundary elements, eddy current problem, finite elements.

1. Introduction

Eletromagnetic forming of metalic sheets relies on generating pulses of eddy currents, which are imposed by a surrounding coil. This gives rise to the Lorentz forces that are pushing the metalic sheet against a form. In order to analyze and later optimize this metalurgical process we shall model the transient eddy current problem and propose a numerical method that gives accurate enough results. This is the aim of the present paper. Other parts of the model such as contact mechanics, plasticity, and eventually thermal distribution shall be treated elsewhere.

We consider a domain $\Omega^{\text{int}} \subset \mathbb{R}^3$ occupied by the metalic sheet and the exterior $\Omega^{\text{ext}} := \mathbb{R}^3 \setminus \overline{\Omega^{\text{int}}}$. The transient eddy current problem reads as follows: for $i \in \{\text{int}, \text{ext}\}$ and $(x,t) \in \Omega^i \times \mathbb{R}_+$ compute the distributions of the magnetic strength density and the electric intensity

$$\mathbf{H}(x,t) := \begin{cases} \mathbf{H}^{\text{int}}(x,t) & x \in \Omega^{\text{int}}, \\ \mathbf{H}^{\text{ext}}(x,t) & x \in \Omega^{\text{ext}}, \end{cases}$$

$$\mathbf{E}(x,t) := \begin{cases} \mathbf{E}^{\text{int}}(x,t) & x \in \Omega^{\text{int}}, \\ \mathbf{E}^{\text{ext}}(x,t) & x \in \Omega^{\text{ext}}, \end{cases} \tag{1}$$

respectively, that satisfy the low frequency case of Maxwell's equations

$$\begin{aligned} \tfrac{\partial}{\partial t}\mathbf{H}^i(x,t) &+ \tfrac{1}{\mu_0}\,\mathbf{curl}\,\mathbf{E}^i(x,t) &= \mathbf{0}, \\ \mathbf{curl}\,\mathbf{H}^i(x,t) &- \sigma^i(x)\,\mathbf{E}^i(x,t) &= \mathbf{J}^i(x,t), \\ \operatorname{div}\mathbf{H}^i(x,t) && = 0, \\ & \operatorname{div}\mathbf{E}^i(x,t) &= 0. \end{aligned} \tag{2}$$

Here, $\mu_0 > 0$ is the permeability of air, $\sigma^{\text{int}} > 0$ is the conductivity of the metallic sheet, $\sigma^{\text{ext}} := 0$, $\mathbf{J}^{\text{ext}}$ is the impressed current density, and $\mathbf{J}^{\text{int}} := \mathbf{0}$. The equations are completed by the transmission conditions: for

$(x, t) \in \partial\Omega^{\text{int}} \times \mathbb{R}_+$

$$\begin{aligned} \mathbf{H}^{\text{ext}}(x,t) - \mathbf{H}^{\text{int}}(x,t) &= \mathbf{0}, \\ \mathbf{E}^{\text{ext}}(x,t) - \mathbf{E}^{\text{int}}(x,t) &= \mathbf{0}, \end{aligned} \quad (3)$$

the decay conditions: for $|x| \to \infty$ and $t \in \mathbb{R}_+$

$$|\mathbf{E}^{\text{ext}}(x,t)|, \; |\mathbf{H}^{\text{ext}}(x,t)| \to 0, \quad (4)$$

and the initial conditions: for $x \in \mathbb{R}^3$

$$\mathbf{H}(x,0) = \mathbf{E}(x,0) = \mathbf{0}. \quad (5)$$

There are several approaches to formulate the model in the sense of distributions, which is the best-known concept allowing for geometrical as well as material singularities or jumps. Basically, there are potential-based formulations [11] and [12], magnetic-field-based (H-based) formulations [6] and [15] and electric-field-based (E-based) formulations [1], [2], [3], [4], [5], [10] and [17]. We prefer the latter approach, with which we are experienced [13] and [14].

The rest of the paper is organized as follows: In Sec. 2. we present the E-based variational formulation in Ω^{int} and a finite element discretization. In Sec. 3. we recall Stratton-Chu representation and boundary element method (BEM) in the exterior. Section 4. is devoted to Hiptmair's symmetric FEM-BEM coupling. In Sec. 5. numerical results are presented. We give conclusions in Sec. 6.

2.   E-Based FEM

After applying **curl** to the first equation in Eq. (2), $\partial/\partial t$ to the second one, and adding both we arrive (up to μ_0) at the E-based formulation of Eq. (2): for $(x, t) \in \Omega^i \times \mathbb{R}_+$

$$\sigma^i \tfrac{\partial}{\partial t}\mathbf{E}^i(x,t) + \tfrac{1}{\mu_0}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^i(x,t) = -\tfrac{\partial}{\partial t}\mathbf{J}^i(x,t),$$

$$\operatorname{div}\mathbf{E}^i(x,t) = 0. \quad (6)$$

A variational formulation of Eq. (6) was introduced and analyzed in [4]. It reads as follows: find $\mathbf{E}^{\text{int}} \in \mathcal{V} := L^2\big((0,T), \mathbf{H}(\mathbf{curl}; \Omega^{\text{int}})\big) \cap H^1\big((0,T), \mathbf{H}^{-1}(\mathbf{curl}; \Omega^{\text{int}})\big)$ such that

$$\underbrace{\mu_0\,\sigma \int_{\Omega^{\text{int}}} \frac{\partial}{\partial t}\mathbf{E}^{\text{int}}(x,t) \cdot \mathbf{v}(x)\,\mathrm{d}x}_{=:\langle \mathbf{M}(\partial_t\,\mathbf{E}^{\text{int}}), \mathbf{v}\rangle}$$

$$+ \underbrace{\int_{\Omega^{\text{int}}} \mathbf{curl}\,\mathbf{E}^{\text{int}}(x,t) \cdot \mathbf{curl}\,\mathbf{v}(x)\,\mathrm{d}x}_{=:\langle \mathbf{A}(\mathbf{E}^{\text{int}}), \mathbf{v}\rangle}$$

$$- \underbrace{\int_{\Gamma} \boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\text{int}}(x,t) \cdot \boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{v}(x)\,\mathrm{d}S(x)}_{=:\langle \mathbf{f}(\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\text{int}}), \mathbf{v}\rangle} = 0, \quad (7)$$

for all $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega^{\text{int}})$. Here, $\Gamma := \partial\Omega^{\text{int}}$, $\mathbf{n}$ is the outer unit normal vector to Ω^{int}, and we define the following Dirichlet and Neumann traces, respectively,

$$\begin{aligned} \boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{v}(x) &:= \mathbf{n}(x) \times (\mathbf{v}(x) \times \mathbf{n}(x)), \\ \boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{u}(x) &:= \mathbf{curl}\,\mathbf{u}(x) \times \mathbf{n}(x). \end{aligned} \quad (8)$$

The formulation is completed by a boundary condition and the initial condition.

We approximate Sobolev space $\mathbf{H}(\mathbf{curl}; \Omega^{\text{int}})$ using the lowest-order Nedelec-I finite elements [16]. We search for a piecewise polynomial approximation

$$\mathbf{E}^{\text{int}}(x,t) \approx \sum_{i=1}^{n} e_i(t)\boldsymbol{\varphi}_i(x), \quad (9)$$

and arrive at the system of ordinary differential equations (ODEs): for $t \in \mathbb{R}_+$

$$\begin{aligned} \mathbf{M}\,\mathbf{e}^{\text{int}\,\prime}(t) + \mathbf{A}\,\mathbf{e}^{\text{int}}(t) + \mathbf{f}(\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\text{int}})(t) &= \mathbf{0}, \\ \mathbf{e}^{\text{int}}(0) &= \mathbf{0}, \end{aligned} \quad (10)$$

where we denote by $\mathbf{M}$ and $\mathbf{A}$ the so-called conductivity matrix and permittivity matrix, respectively. Yet $\mathbf{f}$ is to be specified. We can solve the ODEs analytically as far as we are able to find the eigenvalues λ and the eigenvectors of the matrix pencil $\mathbf{A} - \lambda\,\mathbf{M}$. This can be typically done for $n \leq 10^3$. Otherwise, we have to employ a time-integration scheme.

Note that in a pure FEM the domain Ω^{int} has to be actually extended by a large portion of Ω^{ext} so that the support of $\mathbf{J}$ is included. On the boundary of this extended domain the electric field is assumed to vanish, thus, the boundary term $\mathbf{f}$ disappears. On the right-hand side there is an extra term related to $-\partial_t\mathbf{J}$.

3.   E-Based BEM

In the exterior domain we follow the approach of Hiptmair [10]. We employ Stratton-Chu representation formula: for $(x, t) \in \Omega^{\text{ext}} \times \mathbb{R}_+$

$$\begin{aligned} \mathbf{E}^{\text{ext}}(x,t) = \mathbf{W}(\boldsymbol{\gamma}_{\mathrm{D}}\mathbf{E}^{\text{ext}}(y,t))(x) \\ - \widetilde{\mathbf{V}}(\boldsymbol{\gamma}_{\mathrm{N}}\mathbf{E}^{\text{ext}}(y,t))(x) + \mathbf{N}(-\mu_0\,\mathbf{J}'_t(y,t))(x), \end{aligned} \quad (11)$$

where

$$\widetilde{\mathbf{V}}(\boldsymbol{\lambda}(y))(x) := \int_{\Gamma} \boldsymbol{\lambda}(y)\,\frac{1}{4\pi|x-y|}\,\mathrm{d}S(y), \quad (12)$$

is the vectorial single-layer operator,

$$\mathbf{W}(\mathbf{u}(y))(x) := \widetilde{\mathbf{V}}(\mathbf{n}(y) \times \mathbf{u}(y))(x), \quad (13)$$

is the Maxwell double-layer operator, and

$$\mathbf{N}(\mathbf{g}(y))(x) := \int_{\Omega^{\text{ext}}} \mathbf{g}(y)\,\frac{1}{4\pi|x-y|}\,\mathrm{d}S(y), \quad (14)$$

is the Newton potential (Biot-Savart law). Note that in general there are two additional terms in the formula, which vanish in our formulation.

Applying $\boldsymbol{\gamma}_{\mathrm{D}}$ to the Stratton-Chu formula leads to the first-kind boundary integral equation: for $(x,t) \in \Gamma \times \mathbb{R}_+$

$$\underbrace{\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{ext}} - \boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{W}(\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{ext}})}_{=:-\mathbf{B}(\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{ext}})(x,t)} + \underbrace{\boldsymbol{\gamma}_{\mathrm{D}}\,\widetilde{\mathbf{V}}(\boldsymbol{\gamma}_{\mathrm{N}}\mathbf{E}^{\mathrm{ext}})}_{=:\mathbf{V}(\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\mathrm{ext}})(x,t)}$$
$$= \underbrace{\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{N}(-\mu_0\,\mathbf{J}'_t)}_{=:\mathbf{c}(x,t)}. \quad (15)$$

Applying $\boldsymbol{\gamma}_{\mathrm{N}}$ to Stratton-Chu formula gives rise to the second-kind boundary integral equation: for $(x,t) \in \Gamma \times \mathbb{R}_+$

$$\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\mathrm{ext}} = \underbrace{\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{W}(\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{ext}})(x,t)}_{=:-\mathbf{D}(\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{ext}})(x,t)} - \underbrace{\boldsymbol{\gamma}_{\mathrm{N}}\,\widetilde{\mathbf{V}}(\boldsymbol{\gamma}_{\mathrm{N}}\mathbf{E}^{\mathrm{ext}})}_{=\mathbf{B}^T(\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\mathrm{ext}})(x,t)}$$
$$+ \underbrace{\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{N}(-\mu_0\,\mathbf{J}'_t)}_{=:\mathbf{b}(t)}. \quad (16)$$

The boundary integral equations Eq. (15) and Eq. (16) are again understood in the Sobolev variational framework [7] and [8], which allows a stable boundary element discretization. The discrete space consists of tangential traces of Nedelec-I elements [16], the so-called stream functions.

4. FEM-BEM Coupling

We need FEM to properly model the transient behaviour of eddy currents. However, FEM can approximate decay condition Eq. (4) only at the high cost of additional volume discretization of a large portion of Ω^{ext}. On the other hand, BEM models the decay condition by definition, but it suffers from modelling of transient fields in Ω^{int}. Fortunately, there is a natural coupling of FEM and BEM, cf. [9]. It allows us to get rid of the unknown Neumann boundary data $\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\mathrm{int}}$ in Eq. (7).

From Eq. (15) we can eliminate the Neumann data of the exterior field

$$\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\mathrm{ext}} = \mathbf{V}^{-1}\left(\mathbf{c} + \mathbf{B}(\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{ext}})\right). \quad (17)$$

Plugging the latter to Eq. (16) we arrive at a boundary integral equation with the exterior Steklov-Poincare (Dirichlet-to-Neumann) operator $\mathbf{S}$

$$\boldsymbol{\gamma}_{\mathrm{N}}\,\mathbf{E}^{\mathrm{ext}} =$$
$$-\underbrace{\left(\mathbf{D} + \mathbf{B}^T\,\mathbf{V}^{-1}\,\mathbf{B}\right)}_{=:\mathbf{S}}\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{ext}}\underbrace{-\mathbf{b} - \mathbf{B}^T\,\mathbf{V}^{-1}\,\mathbf{c}}_{=:\mathbf{d}}. \quad (18)$$

Now the latter and transmission condition Eq. (3) replaces the boundary term in Eq. (7)

$$\langle\mathbf{M}(\partial_t\mathbf{E}^{\mathrm{int}}), \mathbf{v}\rangle + \langle\mathbf{A}(\mathbf{E}^{\mathrm{int}}), \mathbf{v}\rangle + \langle\mathbf{S}(\boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{E}^{\mathrm{int}}), \boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{v}\rangle_\Gamma$$
$$= -\langle\mathbf{d}, \boldsymbol{\gamma}_{\mathrm{D}}\,\mathbf{v}\rangle_\Gamma. \quad (19)$$

Finally, we employ the finite element discretization Eq. (9) and arrive at ODEs: for $t \in \mathbb{R}_+$

$$\mathbf{M}\,\mathbf{e}^{\mathrm{int}\prime}(t)) + \underbrace{\left(\mathbf{A} + (\mathbf{I}_\Gamma)^T\,\mathbf{S}\,\mathbf{I}_\Gamma\right)}_{=:\mathbf{K}}\mathbf{e}^{\mathrm{int}}(t) = -\mathbf{d}(t),$$
$$\mathbf{e}^{\mathrm{int}}(0) = \mathbf{0}, \quad (20)$$

where $\mathbf{I}_\Gamma$ is the restriction to the boundary degrees of freedom (identity matrix completed by zeros). The resulting FEM-BEM system of ODEs can be analytically integrated in time

$$\mathbf{e}^{\mathrm{int}}(t) = -\sum_{i=1}^{n}\left(\int_0^t \mathbf{v}_i \cdot \mathbf{d}(\tau)\,\mathrm{e}^{-\lambda_i\,(t-\tau)}\,\mathrm{d}\tau\right)\mathbf{v}_i, \quad (21)$$

where

$$\mathbf{K}\,\mathbf{v}_i = \lambda_i\,\mathbf{M}\,\mathbf{v}_i, \quad \|\mathbf{v}_i\|_{\mathbf{M}} = 1. \quad (22)$$

5. Numerical Results

We consider an axisymmetric setup of a coil and an aluminium plate disc, see Fig. 1. The radius of the disc is 8 cm and the disc is 2 mm thin. It is placed 2 mm above the coil. The coil is modeled by 3 line circular turns of radii $r_k \in \{2.1, 3.7, 5.3\}$ cm. Hence, we replace the Newton potential by the following dimensionally-reduced Biot-Savart law:

$$\mathbf{N}(x) := -\frac{\mu_0\,I}{4\pi}\sum_{k=1}^{3}\int_0^{2\pi}\frac{(-\sin t, \cos t, 0)}{\|x - r_k\,(\cos t, \sin t, 0)\|}\,r_k\,\mathrm{d}t. \quad (23)$$

The excited current pulse has the amplitude $I := 100$ kA. The shape $g(t)$ is half of the sine at frequency $f := 8.33$ kHz,

$$g(t) := \begin{cases} \sin(2\pi\,f\,t), & t \le \frac{1}{2f}, \\ 0, & t \ge \frac{1}{2f}. \end{cases} \quad (24)$$

We employ the particular solution approach: find $\mathbf{E}^{\mathrm{int}}(x,t) = \mathbf{E}_0^{\mathrm{int}}(x,t) + g'(t)\,\mathbf{N}(x)$ so that $\mathbf{E}_0^{\mathrm{int}} \in \mathcal{V}$, $\mathbf{E}_0^{\mathrm{int}}(x,0) = \mathbf{0}$, and

$$\langle(\partial_t + \mathbf{K})\mathbf{E}_0^{\mathrm{int}}, \mathbf{v}\rangle = -g''(t)\,\langle\mathbf{N}, \mathbf{v}\rangle \quad \forall\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega^{\mathrm{int}}),$$

which is the counterpart to Eq. (19).

In Fig. 2 we depict a comparison of eddy current distribution and Lorentz forces computed by FEM and FEM-BEM methods. The numbers of uknowns were 9751 in case of FEM and 800 in case of FEM-BEM. The difference in the Lorentz force magnitudes is shown in Fig. 3. The FEM-BEM results are, by definition, more precise.
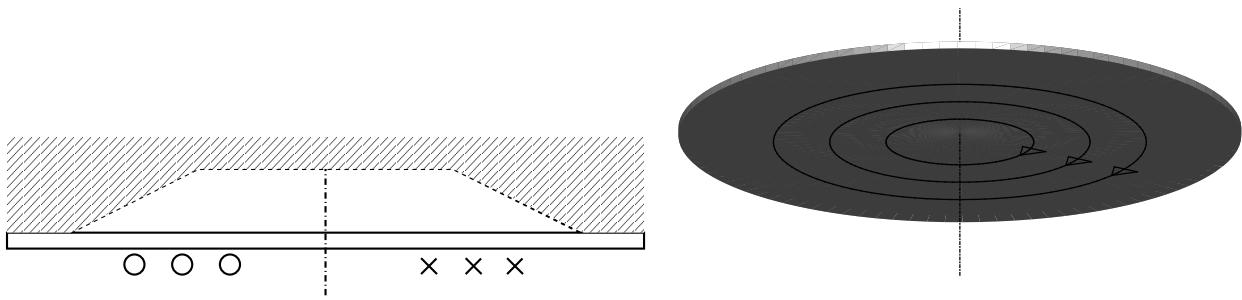
Fig. 1: Geometry of the example. On the left figure a sketch of the device is depicted. The metalic plate (solid rectangle) is pushed against the form (hatched object on the top). Outward and inward orientation of the currents in the three circular turns is depicted with circles and crosses, respectively. The right figure shows the situation in 3D.
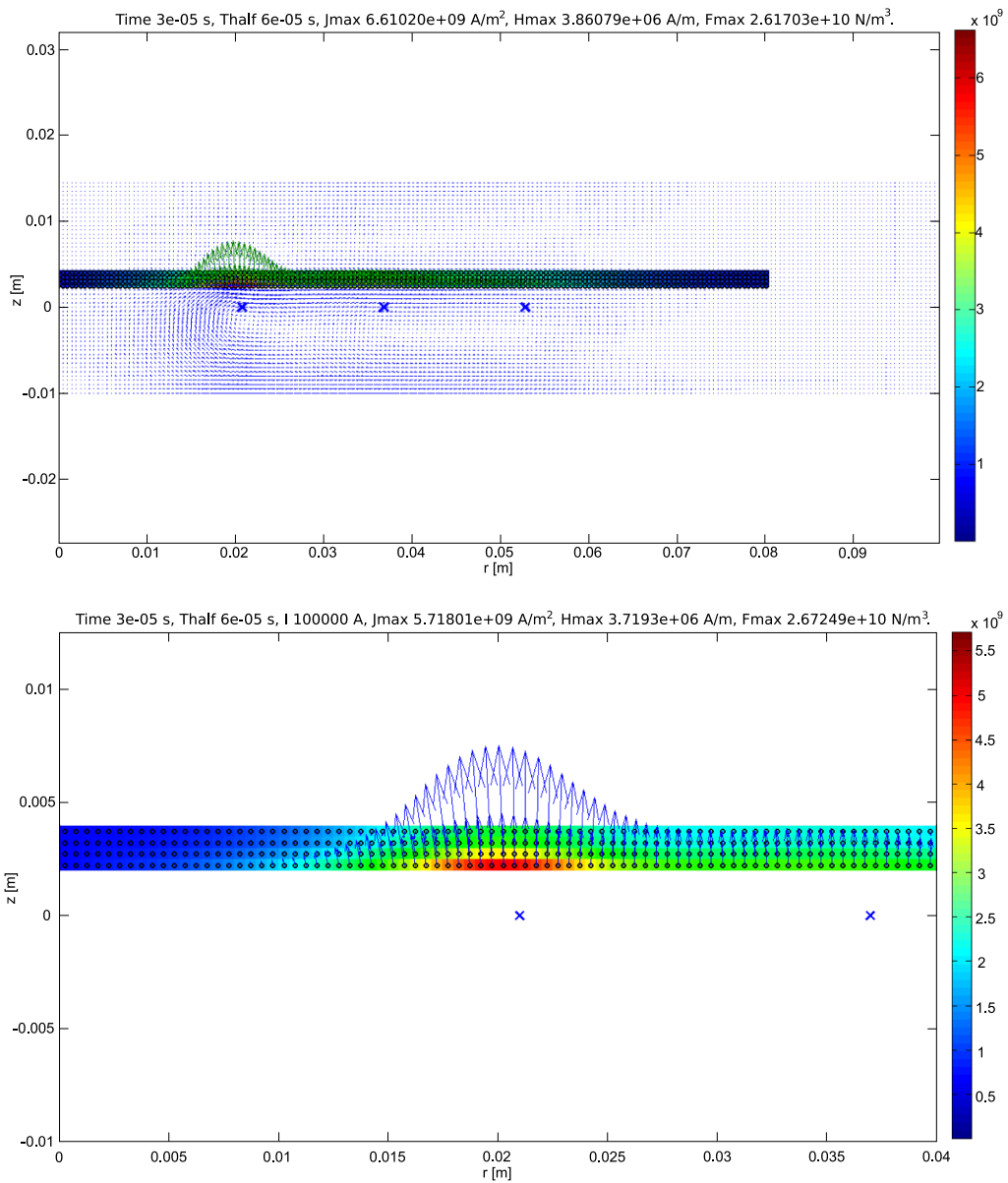
Fig. 2: Comparison of eddy current distributions and Lorentz forces calculated by FEM (top figure) and FEM-BEM (bottom figure) at the half-period of the pulse.
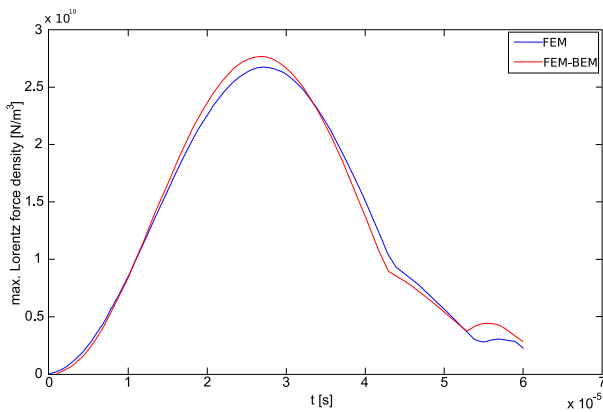
Fig. 3: Evolution of the maximal Lorentz force density computed by FEM (blue line) and FEM-BEM (red line) methods.

6. Conclusion

We presented a coupling of FEM and BEM for solution of transient eddy current problems that arise in the process of electromagnetic forming of metalic sheets. In our forthcoming work we shall complete the model with contact mechanics, plasticity, and thermal field distribution.

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