Comparison of Fractional Order Modelling and Integer Order Modelling of Fractional Order Buck Converter in Continuous Condition Mode Operation

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Abstract. *Producing a mathematical model with high accuracy is the first and important step in control of systems. Nowadays fractional calculus has been in the spotlight and it has a lot of application especially in control engineering. Fractional modelling on one of the conventional converters is done in this paper. Fractional state space model and related fractional transfer functions for a fractional DC/DC Buck converter is established and achieved results are compared to integer order models. At the end of this paper Oustaloup's recursive approximation is introduced and imposed for one of gathered fractional transfer function.*

Keywords

Approximation, averaged circuit model, buck converter, fractional calculus, fractional transfer function.

1. Introduction

The irregular pollution of fossil fuel energy and climate changing with greenhouse gases have put renewable energy sources in the spotlight [1]. Wind turbines in both offshore and onshore are one of the most impressive approach for producing energy with renewable sources [1]. Standalone wind turbine systems are recommended and useful in remote area and other locations with suitable potential of wind speed. In standalone systems, battery absorbs the produced electricity by wind turbine and charge control tasks to ensure stability and reach MPPT [2]. Full-power converters mostly with PMSG are used in variable speed wind turbines [3]. In stand-alone systems DC/DC converters are one of the most important component which are playing and vital role.

The main purpose of DC/DC power electronic converters which are applied widely in switching power supplies and dc motor's drives is to control the amplitude of output voltage and current. In many cases the input power is unregulated and control scheme is essential to yield a regulated and appropriate output power [4]. Most of the engineers and researches have considered integer order models for all systems including power electronic systems and devices which are made up of fractional components in nature [5]. During the last few decades' fractional calculus has opened new horizons in all engineering branches. It seems necessary to replace integer order modeling analysis and modeling by fractional order. The conventional wisdom is that the electrical elements like inductors and capacitors are integer order in nature but the reality is different. Lots of researches have been done in order to prove the fractionality of electrical elements. In [6], [7], [8] and [9] Wesrelund et al. measured practically the fractional order of capacitors and inductors with different dielectrics and core coil. For instance, at 1 kHz frequency and room temperature. They found that the fractional order is 0.9776 for capacitor with polyvinylidenefluoride as dielectric or 0.99911 for capacitor with polysulfide as dielectric and some other capacitor with different dielectric was measured and also 0.97 is the amount of fractionality which gathered for an inductor with air core coil. The fractional property is not terminating to electrical elements and it's possible to expand it to all systems. Fractional order modeling is more accurate than integer order model in many real dynamical circuits [7]. The common integer order models for many electrical circuits are accurate enough because the inductors and capacitors which exist in markets are in fractional order of near 1but for circuits which are made up of fractional components it's essential to describe by fractional order models although in general fractional order modeling can cause more accuracy even for systems with integer order components [8]. It's proved that all systems have fractional characteristic in nature but they are different in amount of being fractional thus integer order models can describe features of many systems which have less fractionalities [9]. In this paper an assumptive fractional order Buck converter is modeled and frequency analysis upon achieved fractional order transfer function is done. Fractional model for Buck converter is compared with integer order model and the gathered results are shown in figures. The perturbation of inductor current and capacitor voltage is computed for fractional Buck converter and finally Oustaloup's recursive approximation is done for one of gathered transfer function.

2. Fractional Calculus

More than three centuries fractional calculus was considered as a theoretical field without any practical applications but during the last three decades this mathematical branch has become more common and useful in lots of sciences and engineering fields such as reaction-diffusion system, electrical circuits, rotor bearing system, finance system, biological system, thermoelectric system, and so on [11]. Perhaps in near future conventional calculus replace by fractional calculus because of its ability to expand usual controllers in order to achieve better performance [12]. Over the last few years the applications of fractional control spread out due to its robust performance [13]. Some fractional definitions which are used in this paper are presented in following equations.

Definition 1 *Rieman-Liouville fractional order integral is defined as bellow,*

$$I_c^{\alpha}F(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_c^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad t > c, \quad \alpha \in R^+, \tag{1}$$

$\Gamma(.)$ *is the Gamma function and* α *is the order of fractionality.*

Definition 2 *Most of the definition in fractional calculus relies on Gamma function which is well known [13]. The definition of Gamma function for real positive number is presented in Eq. (2) [18].*

$$\Gamma(n) = \int_0^{\infty} e^{-u} u^{n-1} du. \tag{2}$$

Remark 1 *If* n *belongs to natural numbers, Eq. (2) will change into factorial form [13].*

$$\Gamma(n) = (n-1)!, \quad n \in N. \tag{3}$$

Note that, substituting α by R^- in order to reach fractional order differential operator is not allowable.

Definition 3 *Rieman-Liouville definition for the fractional order derivative of order* $\alpha \in R^+$ *has the following form,*

$$\begin{aligned} {}_RD^{\alpha}f(t) &\triangleq D^m I^{m-\alpha} f(t) = \\ &= \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m-1}} d\tau\right], \end{aligned} \tag{4}$$

where $m-1 < \alpha < m$, $m \in N$ *and* R *is the representativeness of Rieman-Liouville definition.*

Definition 4 *Caputo derivative is defined as Eq. (5).*

$$\begin{aligned} {}_cD^{\alpha}f(t) &\triangleq I^{m-\alpha} D^m f(t) = \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \end{aligned} \tag{5}$$

where $m-1 < \alpha < m$, $m \in N$ *and* C *shows the Caputo definition.*

Definition 5 *Laplace transform is the next operator which is used in this paper. It is given for Caputo fractional-order derivative as defined in Eq. (6).*

$$\ell\left[{}_cD^{\alpha}f(t)\right] = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0). \tag{6}$$

Definition 6 *Another significant definition is Mittag-Leffler function which is playing the role of an exponential function in integer order response [10]. One-parameter and two parameters Mittag-Leffler function is introduced in following Eq. (7).*

$$\begin{aligned} E_{\alpha}(t) &= \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}, \\ E_{\alpha,\beta}(t) &= \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \\ &\Re(\alpha) > 0, \ \Re(\beta) > 0, \end{aligned} \tag{7}$$

$E(.)$ *is the Mittag-Leffler function.*

3. State Space Model

Figure 1 shows the main scheme of a DC/DC Buck converter which is controlled by Pulse Width Modulation (PWM) unit.

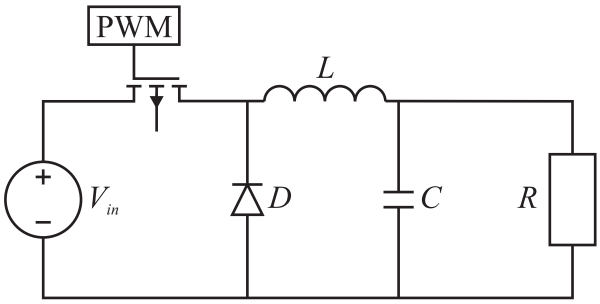

Fig. 1: Shows the circuit of DC/DC Buck converter.

In Eq. (8) and Eq. (9) the well-known equations which present the relationship between voltage and current of inductor and capacitor in fractional order condition is expressed respectively [8].

$$V_L(t) = L\frac{d^\alpha I_L(t)}{dt^\alpha}. \quad (8)$$

$$I_C(t) = C\frac{d^\beta V_C(t)}{dt^\beta}. \quad (9)$$

In which α, β indicates the fractional order of inductor and capacitor respectively. Capacitor voltage and inductor current consider as state space parameters and input voltage is allocated as input vector.

$$X = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}, \quad U = [V_{in}(t)]. \quad (10)$$

By using the following steps averaged model will be established. Two modes consider for DC/DC Buck converter, switch on and switch off mode.

3.1. Switch ON Mode

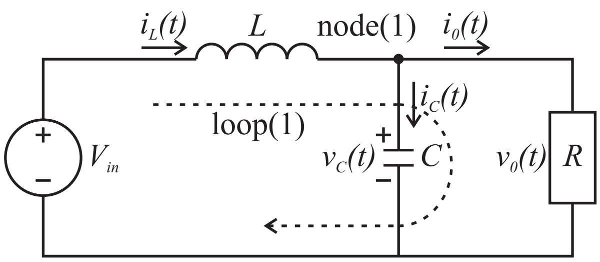

Fig. 2: Buck converter circuit in switch on mode.

Equation (11) is achieved by imposing voltage rule on circuit which is drawn in Fig. 2.

$$\begin{gathered} L\frac{d^\alpha i_L(t)}{dt^\alpha} = v_{in}(t) - v_0(t) \\ \rightarrow \frac{d^\alpha i_L(t)}{dt^\alpha} = \frac{v_{in}(t)}{L} - \frac{v_0(t)}{L}, \end{gathered} \quad (11)$$

Equation (12) also is the result of imposing current rule on circuit which is drawn in Fig. 2.

$$\begin{gathered} C\frac{d^\beta v_C(t)}{dt^\beta} = i_L(t) - \frac{v_C(t)}{R} \\ \rightarrow \frac{d^\beta v_C(t)}{dt^\beta} = \frac{i_L(t)}{C} - \frac{v_C(t)}{RC}, \end{gathered} \quad (12)$$

State space model can be written as bellow,

$$\begin{bmatrix} \frac{d^\alpha i_L(t)}{dt^\alpha} \\ \frac{d^\beta v_C(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v_{in}(t),$$

$$[v_0(t)] = [0 \quad 1] \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}, \quad (13)$$

therefore

$$\begin{gathered} \mathbf{A}_1 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \ \mathbf{B}_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \\ \mathbf{C}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ \mathbf{D}_1 = 0. \end{gathered} \quad (14)$$

3.2. Switch OFF Mode

In this condition the same procedure as former condition is applied. Diode is passed the current as a short circuit path.

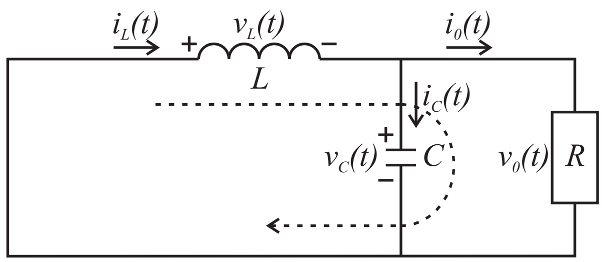

Fig. 3: Buck converter circuit in switch off mode.

Voltage rule:

$$v_C(t) = -L\frac{d^\alpha i_L(t)}{dt^\alpha} \rightarrow \frac{d^\alpha i_L(t)}{dt^\alpha} = -\frac{v_C(t)}{L}. \quad (15)$$

Current rule:

$$i_C(t) = i_L(t) - i_0(t) \rightarrow \frac{d^\beta v_C(t)}{dt^\beta} = \frac{i_L(t)}{C} - \frac{v_C(t)}{RC}. \quad (16)$$

Equation (15) and Eq. (16) is used to reach the state space model of DC/DC Buck converter in switch off mode condition.

$$\begin{bmatrix} \frac{d^\alpha i_L(t)}{dt^\alpha} \\ \frac{d^\beta v_C(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_{in},$$

$$[v_0(t)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + [0] v_{in}. \quad (17)$$

Thus,

$$\mathbf{A}_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (18)$$
$$\mathbf{C}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \mathbf{D}_2 = 0.$$

As can be seen in fractional state space model which was obtained for DC/DC Buck converter, the additional parameters (α, β) indicates the fractional order of inductor and capacitor respectively and in comparison to integer order model it is more complicated but this complexity lead us to more accuracy.

4. Averaged State Space Model of Fractional Order Buck Converter

According to averaging formula, it's possible to substitute averaged value of each parameter in state space model of DC/DC Buck converter.

$$\langle x(t) \rangle_T = \frac{1}{T} \int_t^{t+T} x(\tau) d\tau, \quad (19)$$

where x is an arbitrary variable of the Buck converter.

By averaging a circuit variable over a period of switching, all the high frequency switching harmonics will be removed [7].

$$\frac{d^\alpha \langle x(t) \rangle_T}{dt^\alpha} = \left\langle \frac{d^\alpha x(t)}{dt^\alpha} \right\rangle_T. \quad (20)$$

Equation (20) can be proved easily [7].

Each variable of Buck converter is made up of AC and DC components which are presented in Eq. (21).

$$\begin{aligned} \langle i_L(t) \rangle &= I_L + \hat{i}_L(t), \quad \langle v_C(t) \rangle = V_C + \hat{v}_C(t), \\ \langle v_{in}(t) \rangle &= V_{in} + \hat{v}_{in}(t), \quad \langle d(t) \rangle = D + \hat{d}(t), \end{aligned} \quad (21)$$

where I_L and $\hat{i}_L(t)$ are DC and AC component of inductor current respectively.

It's time to build the averaged state space model of Buck converter by using Eq. (14) and Eq. (18) which depend on switch ON and switch OFF mode.

$$\begin{aligned} d(t) = \frac{T_{on}}{T},\ d'(t) = \frac{T_{off}}{T},\ d(t) + d'(t) = 1, \\ \mathbf{A} = d(t)\mathbf{A}_1 + d'(t)\mathbf{A}_2, \\ \mathbf{B} = d(t)\mathbf{B}_1 + d'(t)\mathbf{B}_2, \\ \mathbf{C} = d(t)\mathbf{C}_1 + d'(t)\mathbf{C}_2. \end{aligned} \quad (22)$$

Thus,

$$\begin{aligned} \mathbf{A} = d(t) \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & \frac{1}{RC} \end{bmatrix} + d'(t) \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & \frac{1}{RC} \end{bmatrix} = \\ = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & \frac{1}{RC} \end{bmatrix}. \end{aligned} \quad (23)$$

B and **C** matrices are gathered as same as **A**.

$$\begin{aligned} \mathbf{B} = d(t) \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} + d'(t) \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{d(t)}{L} \\ 0 \end{bmatrix}, \\ \mathbf{C} = d(t) \begin{bmatrix} 0 & 1 \end{bmatrix} + d'(t) \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \end{aligned} \quad (24)$$

Hence, the averaged state space model can be written as follow according to Eq. (23) and Eq. (24).

$$\begin{aligned} \begin{bmatrix} \frac{d^\alpha i_L(t)}{dt^\alpha} \\ \frac{d^\beta v_C(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \ldots \\ \ldots + \begin{bmatrix} \frac{d(t)}{L} \\ 0 \end{bmatrix} v_{in}(t). \end{aligned} \quad (25)$$

Equations (20) and Eq. (21) are used in the averaged state space model of DC/DC Buck converter.

$$\begin{aligned} \begin{bmatrix} \frac{d^\alpha \langle i_L(t) \rangle}{dt^\alpha} \\ \frac{d^\beta \langle v_C(t) \rangle}{dt^\beta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} \langle i_L(t) \rangle \\ \langle v_C(t) \rangle \end{bmatrix} + \ldots \\ \ldots + \begin{bmatrix} \left\langle \frac{d(t)}{L} \right\rangle \\ 0 \end{bmatrix} \langle v_{in}(t) \rangle. \end{aligned} \quad (26)$$

The averaged value of used variables in Eq. (21) is replaced by their AC and DC components then the Eq. (27) is reached.

$$\begin{bmatrix} \frac{d^\alpha (I_L + \widehat{i}_L(t))}{dt^\alpha} \\ \frac{d^\beta (V_C + \widehat{v}_C(t))}{dt^\beta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \cdots$$
$$\cdot \begin{bmatrix} I_L + \widehat{i}_L(t) \\ V_C + \widehat{v}_C(t) \end{bmatrix} + \begin{bmatrix} \frac{D + \widehat{d}(t)}{L} \\ 0 \end{bmatrix} \left[V_{in} + \widehat{v}_{in}(t) \right]. \tag{27}$$

5. DC Analysis

In DC analysis, AC value of variables will be omitted and each averaged value substitute by its DC component. Due to being zero in Caputo derivative of DC component, the following equation can be achieved.

$$0 = \mathbf{A} \begin{bmatrix} I_L \\ V_C \end{bmatrix} + \mathbf{B}U, \tag{28}$$

$$\begin{bmatrix} I_L \\ V_C \end{bmatrix} = -\mathbf{A}^{-1}\mathbf{B}U,$$
$$\begin{bmatrix} I_L \\ V_C \end{bmatrix} = -\begin{bmatrix} -\frac{L}{R} & C \\ -L & 0 \end{bmatrix} \begin{bmatrix} \frac{D}{L} \\ 0 \end{bmatrix} V_{in}. \tag{29}$$

Finally,

$$\rightarrow \begin{bmatrix} I_L \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{DV_{in}}{R} \\ DV_{in} \end{bmatrix}. \tag{30}$$

6. Small Signal Analysis

By omitting the derivative of DC component of variables and multiplied small signal component in small signal analysis, AC part of equations is derived [15].

$$\frac{d^\alpha \widehat{i}_L(t)}{dt^\alpha} = \frac{1}{L}\left(-\widehat{v}_C(t) + D\widehat{v}_{in}(t) + \widehat{d}V_{in} \right), \tag{31}$$

$$\frac{d^\beta \widehat{v}_C(t)}{dt^\beta} = \frac{1}{C}\left(\widehat{i}_L(t) - \frac{1}{R}\widehat{v}_C(t) \right). \tag{32}$$

Averaged state space model for small signal analysis rewrite as bellow.

$$\begin{bmatrix} \frac{d^\alpha \widehat{i}_L(t)}{dt^\alpha} \\ \frac{d^\beta \widehat{v}_C(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} \widehat{i}_L(t) \\ \widehat{v}_C(t) \end{bmatrix} + \cdots$$
$$\cdots + \begin{bmatrix} \frac{D}{L} \\ 0 \end{bmatrix} \widehat{v}_{in}(t) + \begin{bmatrix} \frac{V_{in}}{L} \\ 0 \end{bmatrix} \widehat{d}(t). \tag{33}$$

In order to analyze nonlinear circuits by linear methods, it's essential to find out the transfer function of nonlinear systems with linearization for linear analysis then achieved results are able to impose on exact and nonlinear models. By assuming zero initial condition and using the definition of Caputo derivative, calculation of transfer functions for DC/DC Buck converter is mentioned as follow.

By imposing Laplace transform to Eq. (31) and Eq. (32) following statements can be reached.

$$s^\alpha \widehat{i}_L(s) = -\frac{1}{L}\widehat{v}_C(s) + \frac{D}{L}\widehat{v}_{in}(s) + \frac{v_{in}}{L}\widehat{d}(s), \tag{34}$$

$$\widehat{i}_L(s) = Cs^\beta \widehat{v}_C(s) + \frac{1}{R}\widehat{v}_C(s). \tag{35}$$

By substituting Eq. (35) into Eq. (34) and equalizing to zero, transfer function of output voltage to input voltage will be gathered.

$$s^\alpha \left(Cs^\beta \widehat{v}_C(s) + \frac{1}{R}\widehat{v}_C(s) \right) = -\frac{1}{L}\widehat{v}_C(s) + \frac{D}{L}\widehat{v}_{in}(s)$$
$$\rightarrow \widehat{v}_C(s) \left(Cs^{\alpha+\beta} + \frac{1}{R}s^\alpha + \frac{1}{L} \right) = \frac{D}{L}\widehat{v}_{in}(s). \tag{36}$$

Thus,

$$G_{\widehat{v}_0 - \widehat{v}_{in}} = \frac{\widehat{v}_0(s)}{\widehat{v}_{in}(s)} |_{\widehat{d}(s)=0} = \frac{D}{LCs^{\alpha+\beta} + \frac{L}{R}s^\alpha + 1}. \tag{37}$$

In order to explain the relationship between output voltage and duty cycle, Eq. (35) is substituted into Eq. (34) and in this part perturbation of input voltage is ignored.

$$Cs^{\alpha+\beta}\widehat{v}_C(s) + \frac{1}{R}s^\alpha \widehat{v}_C(s) = -\frac{1}{L}\widehat{v}_C(s) + \frac{V_{in}}{L}\widehat{d}(s)$$
$$\rightarrow \widehat{v}_C(s) \left(Cs^{\alpha+\beta} + \frac{1}{R}s^\alpha + \frac{1}{L} = \frac{V_{in}}{L}\widehat{d}(s) \right). \tag{38}$$

Therefore,

$$G_{\widehat{v}_0 - \widehat{d}}(s) = \frac{\widehat{v}_0(s)}{\widehat{d}(s)} |_{\widehat{v}_{in}(s)=0} =$$
$$= \frac{V_{in}}{LCs^{\alpha+\beta} + \frac{L}{R}s^\alpha + 1}. \tag{39}$$

From Eq. (35),

$$\widehat{i}_L(s) = Cs^\beta \widehat{v}_C(s) + \frac{1}{R}\widehat{v}_C(s)$$
$$\rightarrow G_{\widehat{v}_C - \widehat{i}_L} = \frac{\widehat{v}_C(s)}{\widehat{i}_L(s)} + \frac{R}{RCs^\beta + 1}. \tag{40}$$

Equation (40) shows the transfer function of output voltage to inductor current.

7. Computation of Inductor Current

As mentioned before, there are two modes operation in DC/DC Buck converter. In this research Continues Condition Mode (CCM) is considered. In this condition the inductor current never touches zero value and always has an amount.

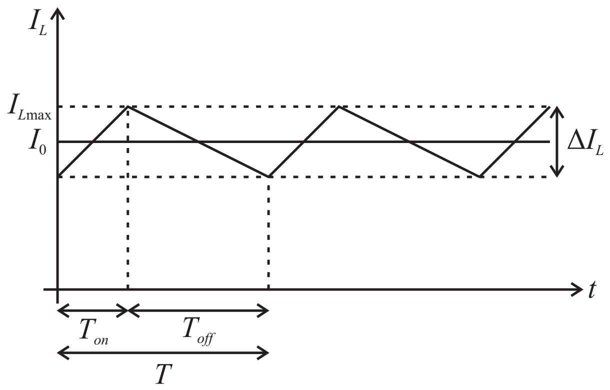

Fig. 4: Inductor current of DC/DC Buck converter.

T indicates the period of switching in ms.

Figure 4 shows the perturbation of inductor current in CCM mode operation. Computational method for finding inductor current is presented as follow. During the period which switch is on (T_{on}), variation of inductor current based on fractional calculus is as bellow:

$$v_L(t) = L\frac{d^\alpha i_L(t)}{dt^\alpha}, \quad v_L(t) = v_{in}(t) - v_0(t). \tag{41}$$

By averaging the both side of Eq. (41) next statement is gathered.

$$\langle v_L(t)\rangle = L\frac{d^\alpha \langle i_L(t)\rangle}{dt^\alpha}, \quad \rightarrow \frac{\langle v_{in}(t) - v_0(t)\rangle}{L} = \frac{d^\alpha \hat{i}_L(t)}{dt^\alpha}. \tag{42}$$

By imposing fractional integrator of order α to both sides of Eq. (42), the next statement will be reached.

$$I_C^\alpha \frac{\langle v_{in}(t) - v_0(t)\rangle}{L} = I^\alpha\left(\frac{d^\alpha \hat{i}_L(t)}{dt^\alpha}\right) = \Delta I_L(t). \tag{43}$$

By assuming that variation of output voltage and input voltage is almost constant and using Caputo definition for integrator, the left side of Eq. (43) can be written as Eq. (44).

$$I_C^\alpha \frac{\langle v_{in}(t) - v_0(t)\rangle}{L} = \frac{1}{\Gamma(\alpha)}\frac{\langle v_{in}(t) - v_0(t)\rangle}{L}\int_0^{DT}(t-\tau)^{\alpha-1}d\tau. \tag{44}$$

DT shows the time which switch is on in (ms) and D is the DC component of duty cycle. In order to get into closed relation for above equation, firstly it's necessary to expand the inner expression of integrator which is used in Eq. (44) by Taylor expansion around zero point then calculate fractional integrator of order α. Matlab is used in this section.

$$\Delta I_L = \frac{(v_{in} - v_0)(DT)^\alpha}{\Gamma(\alpha)L}. \tag{45}$$

Easily can be seen in Fig. 4 the maximum amount of inductor current.

$$I_{L\max} = I_L + \frac{1}{2}\Delta I_L, \quad I_{L\max} = \frac{Dv_{in}}{R} + \frac{1}{2}\frac{(v_{in} - v_0)}{\Gamma(\alpha)L}(DT)^\alpha. \tag{46}$$

Equation (45) and Eq. (46) indicates that the amount of inductor current and its variation depends on not only inductance of inductor but also the amount of fractional order of inductor.

8. Computation of Capacitor Voltage

One of the most important factors for designing Buck converters in order to find the value of inductance and capacitance of inductor and capacitor respectively is the amount of inductor current and capacitor voltage perturbation. Variation of inductor current mentioned in pervious section and capacitor voltage is explained in this section. In Fig. 1 following statement is obvious.

$$i_C(t) = -i_0(t) + i_L(t), \quad \rightarrow \frac{d^\beta v_C(t)}{dt^\beta} = -\frac{v_C(t)}{R} + \frac{i_L(t)}{C}. \tag{47}$$

By using Eq. (47) it's possible to gather the capacitor voltage. Adomian decomposition method is used in [7], [8] and [9] in order to solve the fractional differential equation. This method is a very strong approach for solving the nonlinear equations in case of linear analytically and also it's possible to use it for such Fractional Differential Equation (FDE) [16]. Due to have a solution for Eq. (47) Adomian decomposition method is used. The general form of fractional equation which is solved by Adomian decomposition method is available in Eq. (48).

$$D^\alpha x(t) = Ax(t) + f(t), \quad 0 < \alpha < 1,\ 0 < t < T, \tag{48}$$

$f(.)$ is a function of time and A is a coefficient for x.

Mentioned method in case of Caputo fractional derivative for solving the Eq. (48) is as follow [17].

$$\begin{aligned} x(t) &= x_h(t) + x_p(t), \\ x_h(t) &= E_{\alpha,1}(At^{\alpha})C, \\ x_p(t) &= t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}) \cdot f(t), \end{aligned} \tag{49}$$

where C is the initial condition and E is the Mittag-Leffler function.

Note that the following condition must be satisfied.

$$\exists \;\; \epsilon > 0, \quad M > 0, \quad \beta > -\alpha. \tag{50}$$

Such that, $|f_i(t)| \le Mt^{\beta}$.

During the period which switch is on, Eq. (51) is solved. Note that the Eq. (51) for both periods which switch is on and off is the same because of circuit topology.

$$D^{\beta}V_C(t) = -\frac{V_C(t)}{RC} + \frac{i_L(t)}{C}. \tag{51}$$

Equation (52) shows the numerical solution of Eq. (51) according to Adomian decompodition method which mentioned in Eq. (49).

$$\begin{aligned} V(t) &= E_{\beta,1}\left(-\frac{t^{\beta}}{RC}\right)V_0 + \ldots \\ &\ldots + t^{\beta-1}E_{\beta,\beta}\left(\frac{t^{\beta}}{RC}\right) \cdot \frac{i_L(t)}{C}. \end{aligned} \tag{52}$$

According to Fig. 5 it's possible to find out the output voltage variation [4]. Output voltage and capacitor voltage are the same.

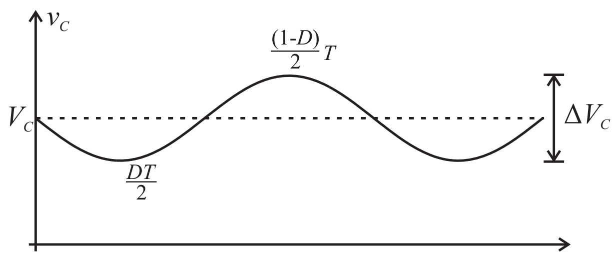

Fig. 5: Capacitor voltage of DC/DC Buck converter.

$$\begin{aligned} V\left(\frac{DT}{2}\right) &= E_{\beta,1}\left(-\frac{\left(\frac{DT}{2}\right)^{\beta}}{RC}\right)V_0 + \left(\frac{DT}{2}\right)^{\beta-1} \\ &\ldots E_{\beta,\beta}\left(-\frac{\left(\frac{DT}{2}\right)^{\beta}}{RC}\right) \cdot \frac{i_L\frac{DT}{2}}{C}. \end{aligned} \tag{53}$$

And,

$$\begin{aligned} V\left(\frac{1-D}{2}T\right) &= E_{\beta,1}\left(-\frac{\left(\frac{1-D}{2}T\right)^{\beta}}{RC}\right)V_0 + \ldots \\ &+ \left(\frac{1-D}{2}T\right)^{\beta-1} + E_{\beta,\beta}\left(-\frac{\left(\frac{1-D}{2}T\right)^{\beta}}{RC}\right)\ldots \\ &\cdot\frac{i_L\left(\frac{1-D}{2}T\right)}{C}. \end{aligned} \tag{54}$$

As it can be seen in Fig. 5 the amount of inductor current in $\frac{DT}{2}$, $\frac{(1-D)}{2}T$ is the same approximately. Therefore,

$$\begin{aligned} \Delta V &= V\left(\frac{1-D}{2}T\right) - V\left(\frac{D}{2}T\right), \\ I\left(\frac{1-D}{2}T\right) &\approx I\left(\frac{D}{2}T\right). \end{aligned} \tag{55}$$

Then,

$$\begin{aligned} \Delta V = &\left[E_{\beta,1}\left(-\frac{\left(\frac{1-D}{2}T\right)^{\beta}}{RC}\right)\ldots \right. \\ &\left. -E_{\beta,1}\left(-\frac{\left(\frac{D}{2}T\right)^{\beta}}{RC}\right)\right]V_0 \cdot \frac{I_0}{C} \cdot \ldots \\ &\left[\left(\frac{1-D}{2}\right)^{\beta-1}E_{\beta,\beta}\left(-\frac{\left(\frac{1-D}{2}T\right)^{\beta}}{RC}\right)\ldots \right. \\ &\left. -\left(\frac{D}{2}T\right)^{\beta-1}E_{\beta,\beta}\left(-\frac{\left(\frac{D}{2}T\right)^{\beta}}{RC}\right)\right]. \end{aligned} \tag{56}$$

9. Approximation

It's possible to describe the dynamical behavior of a fractional order transfer function by an integer order transfer function. Approximation is important because of the numerical solution methods for integer order differential equations are more known and available and also it can found in a lot of common software [13].

9.1. Oustaloup's Recursive Approximation

Mentioned filter is capable to approximate a fractional transfer function with high accuracy. For instance fractional element s^α will be described by $(2N+1)$ integer zeros and poles during the specified frequency interval $[\omega_l, \omega_h]$.

$$s^\alpha = K \prod_{k=-N}^{N} \frac{s+\omega_k'}{s+\omega_k}. \tag{57}$$

Such that,

$$\begin{gathered} \omega_k = \omega_l \left[\frac{\omega_h}{\omega_l}\right]^{\frac{K+N+\frac{1}{2}(1+\alpha)}{2N+1}}, \\ \acute{\omega}_k = \omega_l \left[\frac{\omega_h}{\omega_l}\right]^{\frac{K+N+\frac{1}{2}(1-\alpha)}{2N+1}}, \\ K = (\omega_h)^\alpha. \end{gathered} \tag{58}$$

The order of mentioned filter is $(2N+1)$ and by imposing each input signal to approximated integer transfer function, the output signal will be the response of fractional transfer function [14].

10. Simulation Results

A Buck Converter with following features is assumed:

- $L = 0.236$ mH,
- $C = 47$ mF,
- $R = 0.1\ \Omega$,
- $V_{in} = 28 - 70$ V,
- Output voltage 24 V,
- Switching frequency 30 kHz.

The main reason of current essay is to express the importance of fractional modeling for a fractional system in nature. It's assumed that is possible to have fractional elements like capacitor and inductor although it's not valid in real world up to now. In order to achieve this purpose, Buck converter with above features which used in a 1 kW vertical axis wind turbine assumed but the amount of fractionality for its elements (α, β) is hypothetical values duo to the mentioned Buck converter is made up of integer order components and integer order model has accurate enough for this system. In this section a comparison is done on fractional modeling and integer modeling for studied system. Bode diagram which is mentioned in Fig. 6 is related to transfer function of output voltage to input voltage.

$$\begin{aligned} &G_{\widehat{V}_0-\widehat{V}_{in}} = \frac{0.352}{1.1092e-5s+2.36e-3s^{0.5}+1}, \\ &\alpha = \beta = 0.5. \end{aligned} \tag{59}$$

Equation (59) is gathered by replacing the buck parameters into Eq. (37) and the averaged value for duty cycle is considered as $D = 0.352$.

If the fractional characteristic of mentioned Buck converter is omitted, the quantity of (α, β) will be equaled to 1 and Eq. (59) is converted to an integer order model of Buck converter which is well known equa-

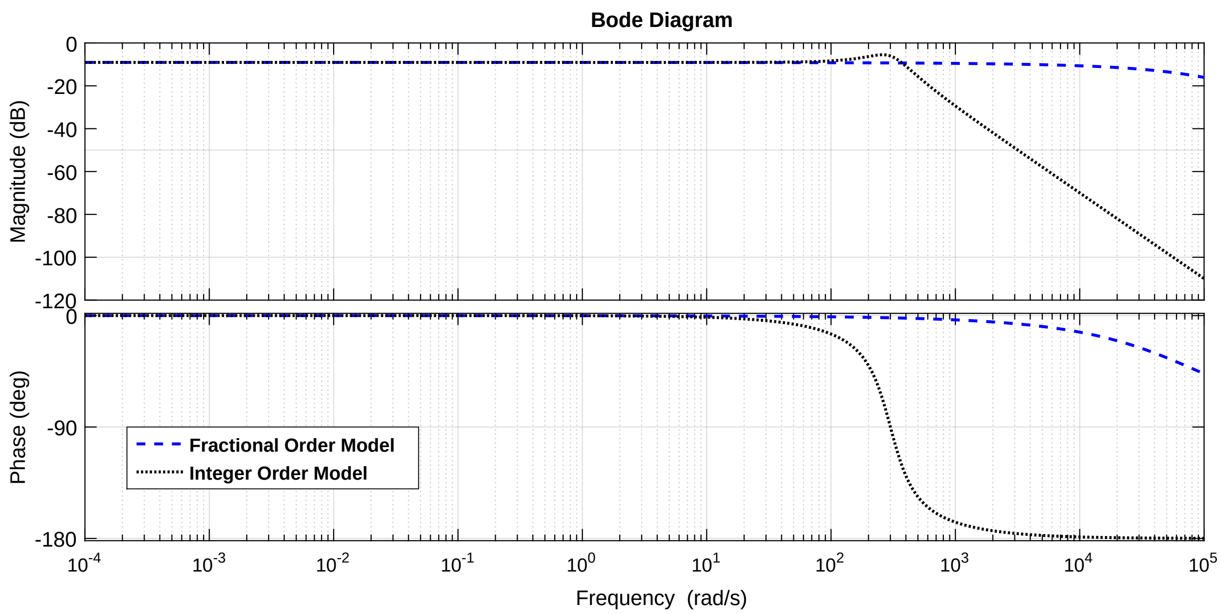

Fig. 6: Frequency response of fractional and integer order model.

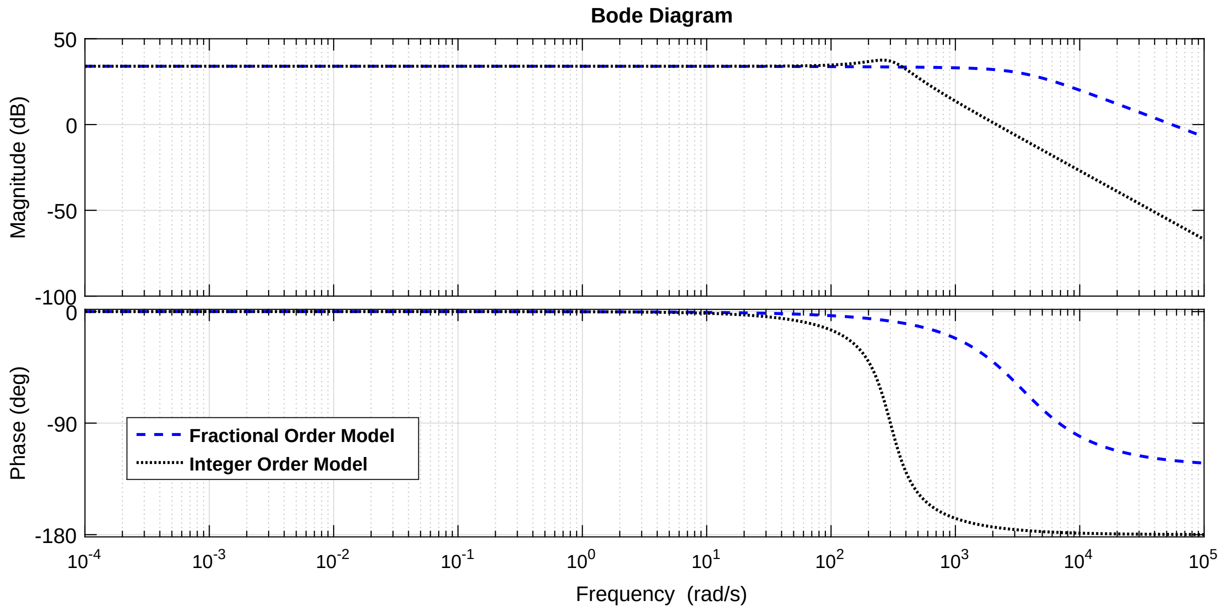

Fig. 7: Frequency response of fractional and integer model.

tion but in this part fractional characteristic is considered for Buck converter and this feature is represented by (α, β).

Fractional operation and simulation can be done easily in Matlab. Some important commands are presented: `Fotf`, `Fomcon`, `Bode`, `Oustapp`, `Plot`.

As can be seen in Fig. 6 integer order modeling for a system which is fractional order in nature (transfer function which mentioned in Eq. (59) causes remarkable difference between real system and integer modeled system.

Frequency analysis has been done on another mentioned transfer function.Assume that the DC value of input voltage is 50 V and fractional order of inductor and capacitor are 0.7 therefore,

$$G_{\widehat{V}_0-\widehat{d}}(s) = \frac{50}{1.1092e-5s^{1.4}+2.36e-3s^{0.7}+1}, \quad \alpha=\beta=0.7. \tag{60}$$

Figure 7 indicates the frequency response of Eq. (60).

The value of (α, β) is hypothetical and just used for simulation and comparison. In order to establishing the fractional order controllers and modeling frac-

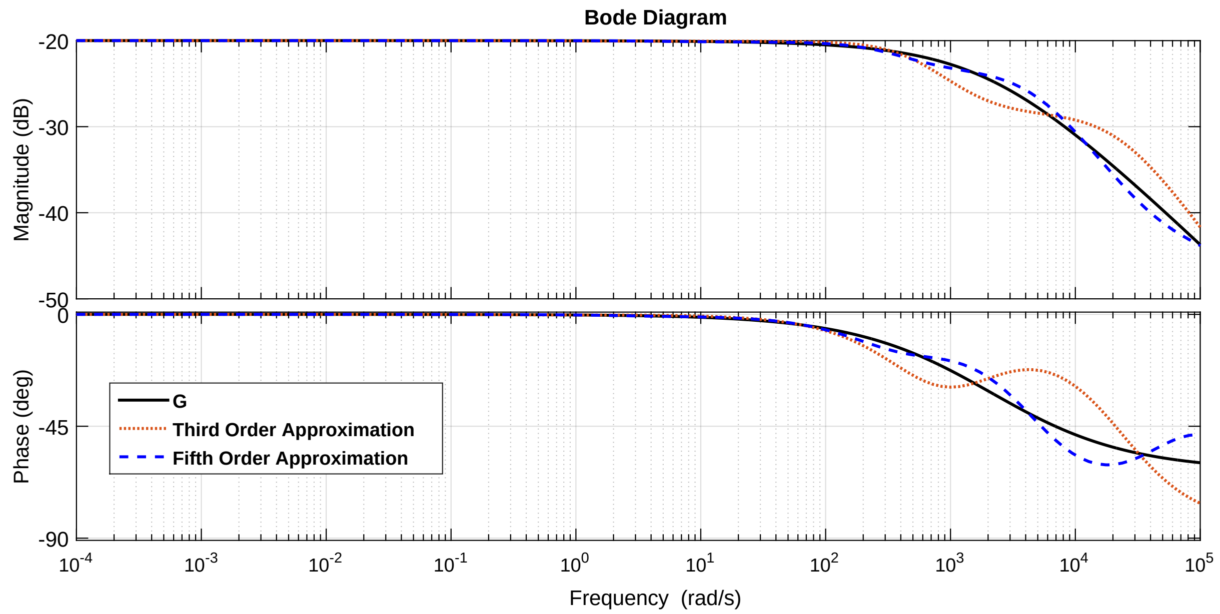

Fig. 8: Frequency response of approximated transfer functions.

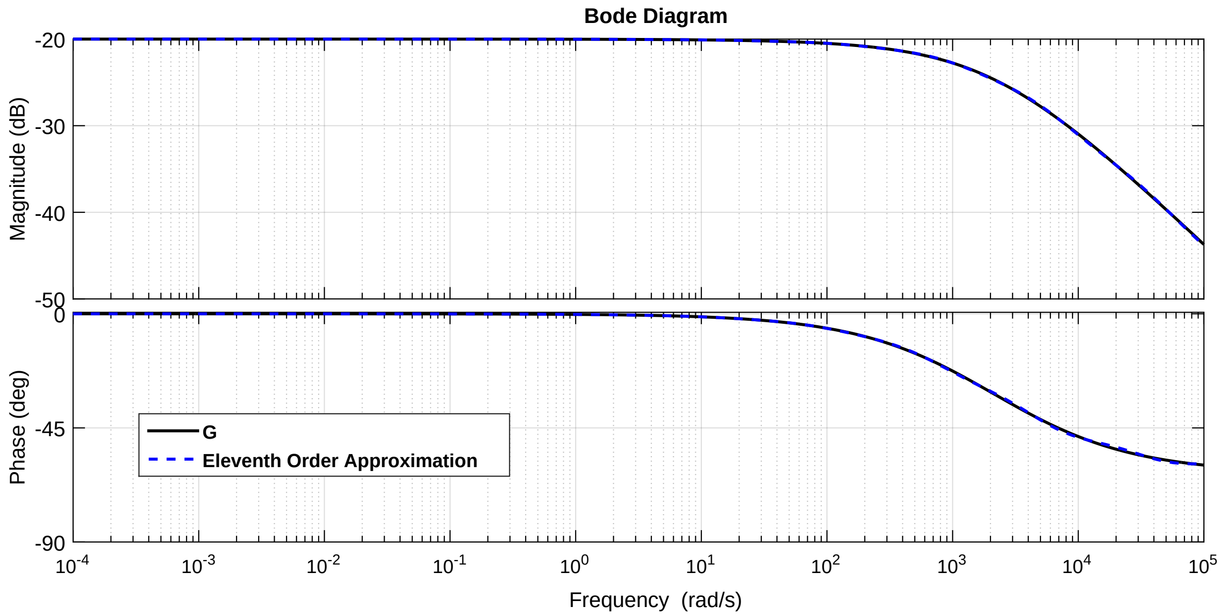

Fig. 9: Frequency response of the most accurate approximation.

tional systems in practical way, it's essential to convert them into integer order transfer function. As mentioned before Oustaloup's recursive approximation can be employed to approximate a fractional order transfer function during a specific frequency interval with $(2N+1)$ zeros and poles. Here $G_{\widehat{V}_C-\widehat{i}_L}$ with $\beta = 0.7$ is considered and approximated by third and fifth integer order transfer function and then their frequency response are compared with the frequency response of fractional transfer function which gathered by using Matlab. As can be seen in Fig. 8 by increasing the order of approximated transfer function the amount of error is decreased.

$$G_{\widehat{V}_C-\widehat{i}_L} = \frac{0.1}{4.7e-3s^{0.7}+1}, \tag{61}$$

$$TF = \frac{2.67e-4(s+3.16e+6)(s+1468)(s+0.6813)}{(s+2.25e+4)(s+554)(s+0.6764)}. \tag{62}$$

Equation (62) shows a third integer order model which approximated for $G_{\widehat{V}_C-\widehat{i}_L}$ in Eq. (61). In practical applications, it's not possible to use fractional equations or controllers and converting into integer order equations and controllers which are the same in characteristics is essential.

In order to have a model with high accuracy an eleventh order approximated transfer function is chosen and is compared by fractional transfer function which is achieved by Matlab.

Figure 9 shows the comparison between eleventh order transfer function which approximated for G and fractional order transfer function (G).

In order to emphasis the importance of fractional order modeling for a fractional order Buck converter, real time simulation was done and compare to integer order model with the same duty cycle. Closed loop transfer function for both Eq. (63) and Eq. (64) was gathered and their step response is ploted.

$$G_{\widehat{V}_0-\widehat{V}_{in}} = \frac{0.352}{1.1092e-5s+2.36e-3s^{0.5}+1}, \quad \alpha = \beta = 0.5. \tag{63}$$

$$G_{\widehat{V}_0-\widehat{V}_{in}} = \frac{0.352}{1.1092e-5s^2+2.36e-3s+1}. \tag{64}$$

Equation (63) and Eq. (64) show the fractional order model and integer order model of fractional order Buck converter with ($\alpha = \beta = 0.5$) respectively.

Input voltage is 70 V for simulation.

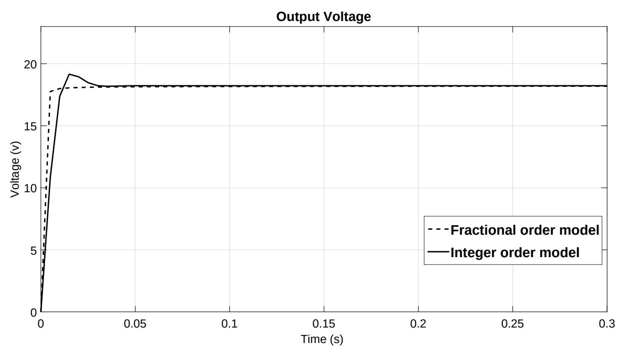

Fig. 10: Fractional order in comparison to integer order model.

Figure 10 shows the closed loop response to constant input voltage and the same duty cycle was imposed to both integer and fractional order model. As mentioned befor, assumed Buck converter was fractional in nature ($\alpha = \beta = 0.5$) then the fractional and integer order model has some differences in output response. Integer order model for fractional Buck converter caused modeling error and its not accurate enough.

11. Conclusion

Although fractional calculus related to a few centuries ago but its useful and important applications are in center of attention especially in engineering nowadays. As mentioned before all systems and components are fractional in nature, but they are different in amount of fractionality [9]. In this paper a DC/DC Buck converter with fractional order components was assumed and frequency analysis along the linear modeling is done. The remarkable difference between integer order model and fractional order model can be seen easily, but for lots of systems integer order modeling is satisfied our purposes and they have enough accuracy because fractionality order of their elements such as capacitors and inductors are near to one. It doesn't mean the fractional order modeling is useless because it can describe the characteristics of systems and circuits with more accuracy but for common and available systems, integer order has enough accuracy. If an extra finesse is required or fractional order elements with fractionality far from one is used, then fractional order modeling will be essential. At the end of research, real time comparison was done for closed loop transfer function of output voltage to input voltage of Buck converter although designing of controller was not the research purpose. Establishing the fractional order model for Buck converter and comparing it to integer order model was the main goal for this research but Fig. 10 shows the non-negligible error in real time simulation for closed loop system which is caused by modeling accuracy. It's obvious that integer order model for a fractional order Buck converter with ($\alpha = \beta = 0.5$) is not a suitable model and makes error.

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